Introduction to the local Langlands correspondence

1. the Weil group

- $\mathbb{F}$: local nonarchimedean field
- $\overline{\mathbb{F}}$: algebraic closure
- $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$: Galois group

$W_{\mathbb{F}}$ is a dense subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, defined by the commutative diagram

inertia group; profinite completion of $\mathbb{Z}$

$$\begin{tikzcd}
1 \arrow[r] & I_{\mathbb{F}} \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_{\mathbb{F}} \arrow[r] \arrow[u, equal] & W_{\mathbb{F}} \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

- $q = \#(\text{residue field of } \mathbb{F})$
- Choose $\mathrm{Fr} \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ with $\mathrm{Fr}(x) = x^q$ on finite fields $\Longrightarrow$

  $W_{\mathbb{F}} = I_{\mathbb{F}} \rtimes \langle \mathrm{Fr} \rangle \cong I_{\mathbb{F}} \rtimes \mathbb{Z}$

  we endow $W_{\mathbb{F}}$ with the product topology

- $\cancel{\|\cdot\| : W_{\mathbb{F}} \to \mathbb{R}_{>0},\ \|i\,\mathrm{Fr}^n\| = q^n}$
- ~~define the Weil-Deligne group as~~

  $\cancel{WD_{\mathbb{F}} := W_{\mathbb{F}} \ltimes \mathbb{C} \quad \text{with} \quad (\gamma, z)(\gamma', z') = (\gamma\gamma', \|\gamma\|^{-1} z z')}$

Artin reciprocity:

$\exists$ natural isomorphism $W_{\mathbb{F}}/\overline{[W_{\mathbb{F}}, W_{\mathbb{F}}]} \cong \mathbb{F}^{\times}$

closure of commutator subgroup

2. Split tori

Def. a $\mathbb{F}$-split torus is an algebraic group T over $\mathbb{F}$, isomorphic to $(\mathbb{F}^\times)^n$ for some $n \in \mathbb{Z}_{\geq 0}$

- $X^*(T)$: lattice of algebraic characters $T \to \mathbb{F}^\times$
- $X_*(T)$: " cocharacters $\mathbb{F}^\times \to T$

Example: T = diagonal torus in $SL_3(\mathbb{F})$

$\mathbb{Z}^2 \cong X^*(T)$ spanned by $\chi_i : \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \mapsto a_i$, with $\chi_1 \chi_2 \chi_3 = 1$

$X_*(T) \cong \mathbb{Z}^2$ spanned by $\check{\alpha} : x \mapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix}$ and $\check{\beta} : x \mapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix}$

- pairing between $X_*(T)$ and $X^*(T)$:
  $\chi \circ \check{\alpha} : \mathbb{F}^\times \to \mathbb{F}^\times$ is of the form
  $x \mapsto x^n$ with $n \in \mathbb{Z}$
  $\longrightarrow \langle \check{\alpha}, \chi \rangle = n$
- $X^*(T)$ and $X_*(T)$ are dual lattices
- $\exists$ natural isomorphism
  $$X_*(T) \otimes_{\mathbb{Z}} \mathbb{F}^\times \longrightarrow T$$
  $$\check{\alpha} \otimes x \longmapsto \check{\alpha}(x)$$

Def. the dual (complex) torus of T is $\hat{T} := X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$

3. the LLC for split tori

- an irreducible representation of T is a character $T \to \mathbb{C}^\times$ which is smooth, i.e. trivial on a compact open subgroup of T

space of irreps

$$\mathrm{Irr}(T) = \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathrm{Hom}(X_*(T)\otimes_{\mathbb{Z}} F^\times, \mathbb{C}^\times)$$

$$\overset{\text{duality}}{\cong} \mathrm{Hom}(F^\times, X^*(T)\otimes_{\mathbb{Z}} \mathbb{C}^\times) = \mathrm{Hom}(F^\times, \check{T})$$

$$\overset{\text{Artin}}{\cong} \mathrm{Hom}(W_F/[W_F, W_F], \check{T}) = \mathrm{Hom}(W_F, \check{T})$$

The natural isomorphism $\mathrm{Irr}(T) \cong \mathrm{Hom}(W_F, \check{T})$ is the LLC for T.

- $\mathcal{O}_F$: integers of F
- $\mathcal{T}(\mathcal{O}_F) \cong \mathcal{O}_F^n$: maximal compact subgroup of $T = \mathcal{T}(F)$

Def. a character of T is unramified if it is trivial on $\mathcal{T}(\mathcal{O}_F)$

$$\begin{array}{ccccccc}
\mathrm{Irr}(T) & \longleftrightarrow & \mathrm{Hom}(W_F, \check{T}) & & & & \\
\cup & & \cup & & & & \varphi \\
X_{nr}(T) & \longleftrightarrow & \{\varphi \in \mathrm{Hom}(W_F, \check{T}) : I_F \subset \ker\varphi\} & & & & \downarrow \\
 & & = \mathrm{Hom}(W_F/I_F, \check{T}) & \cong \mathrm{Hom}(\mathbb{Z}, \check{T}) & \cong & \check{T} & \varphi(Fr)
\end{array}$$

$$\left(\chi \otimes f \mapsto z^{-\langle \chi, \check{\alpha}\rangle \mathrm{val}_F(f)}\right) \longleftrightarrow \check{\alpha} \otimes z$$

4. Split reductive p-adic groups

- G : reductive, connected group over $\mathbb{F}$
  we assume that G is $\mathbb{F}$-split, i.e. ~~...~~ has a maximal torus T that is $\mathbb{F}$-split
- $\check{G}$: complex dual group

Examples:

G	$\check{G}$
$T = X_*(T) \otimes F^\times$	$\check{T} = X^*(T) \otimes \mathbb{C}^\times$
$GL_n(\mathbb{F})$	$GL_n(\mathbb{C})$
$SL_n(\mathbb{F})$	$PGL_n(\mathbb{C})$
$PGL_n(\mathbb{F})$	$SL_n(\mathbb{C})$
$SO_{2n}(\mathbb{F})$	$SO_{2n}(\mathbb{C})$
$SO_{2n+1}(\mathbb{F})$	$Sp_{2n}(\mathbb{C})$
$Sp_{2n}(\mathbb{F})$	$SO_{2n+1}(\mathbb{C})$

G is endowed with the locally compact topology coming from the metric on $\mathbb{F}$

Def. a G-rep (π, V) is smooth if $\forall v \in V$
$\{g \in G : \pi(g)v = v\}$ is an open subgroup of G

G has many compact open subgroups, e.g.
$GL_n(\mathbb{Z}_p)$ and $1 + M_n(p^e\mathbb{Z}_p)$ in $GL_n(\mathbb{Q}_p)$

$\mathrm{Irr}(G) :=$ space of irreducible smooth complex G-reps

5. the LLC for split groups

Def. the Weil–Deligne group of $\mathbb{F}$ is
$WD_{\mathbb{F}} = W_{\mathbb{F}} \ltimes \mathbb{C}$ with $(\gamma,z)(\gamma',z') = (\gamma\gamma', \|\gamma'\|^{-1}z + z')$
where $\| \ \| : W_{\mathbb{F}} \to \mathbb{R}_{>0}$, $\|\underset{\underset{I_F}{\cap}}{i} \cdot Fr^n\| = q^n$

Def. an L-parameter for G is a continuous homomorphism $\varphi : WD_{\mathbb{F}} \to \check{G}$ such that
- $\varphi(W_{\mathbb{F}})$ consists of semisimple elements $\to \varphi(\gamma) \in M_N(\mathbb{C})$ diagonalizable
- $\varphi|_{\mathbb{C}}$ is algebraic and has unipotent image
  $\hookrightarrow \varphi(z) - 1 \in M_N(\mathbb{C})$ nilpotent

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$\Phi(G) :=$ {L-parameters for G} / conjugation by $\check{G}$

special case:
$\Phi(T) = \mathrm{Hom}(W_{\mathbb{F}}, \check{T})$, since $\check{T}$ is commutative and $1 \in \check{T}$ is the only unipotent element

<u>Conjecture</u> (Langlands and others...)
There exists a natural, finite-to-one surjective map

$$\begin{array}{ccc} \mathrm{Irr}(G) & \longrightarrow & \Phi(G) \\ \pi & \longmapsto & \varphi_\pi \end{array}$$

which is functorial with respect to homomorphisms of reductive groups over local fields
The map should be compatible with central characters, parabolic induction, temperedness, square-integrability, ...

6. Unramified L-parameters

Def. an L-parameter $\varphi : WD_F \to \check{G}$ is unramified if $I_F \subset \ker \varphi$.

- $W_F / I_F \cong \langle Fr \rangle \cong \mathbb{Z}$, so $\varphi|_{W_F}$ is determined by $\varphi(Fr)$
- for $z \in \mathbb{C} \subset WD_F$: $\varphi(z) = \varphi(1)^z$
- $\varphi(Fr)\varphi(1)\varphi(Fr)^{-1} = \varphi(Fr \cdot 1 \cdot Fr^{-1}) = \varphi(q) = \varphi(1)^q$

$\Longrightarrow$

Every unramified L-parameter is determined by:
- $s := \varphi(Fr) \in \check{G}$ semisimple
- $N := \varphi(1) \in \check{G}$ unipotent

such that $sNs^{-1} = N^q$

The space of unr. L-parameters is
$\{\text{such } (s, N)\} / \text{conjugacy by } \check{G}$

Example: $G = SL_2(F)$, $\check{G} = PGL_2(\mathbb{C})$
- two unipotent conjugacy classes in $\check{G}$: 1 and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $\check{T}$ = diagonal torus in $PGL_2(\mathbb{C}) = \{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}^\times \}$
- $\{\text{semisimple conj. classes in } \check{G}\} \cong \check{T}/W \cong \mathbb{C}^\times / (z \sim z^{-1})$

  $W = N_{\check{G}}(\check{T}) / \check{T} \cong \{ 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$

Space of unramified L-parameters:
$\{ (s, 1) : s \in \check{T}/W \} \cup \{ (s = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \}$

7. the unramified principal series of $SL_2(\mathbb{F}) = G$

- $T :=$ diagonal torus $= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in F^\times \right\}$
- $B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in F^\times, b \in F \right\}$ standard Borel subgroup
- $U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}$ unipotent radical of B

$$\begin{array}{ccc} X_{nr}(T) & \longleftrightarrow & T^\vee \\ & & \cup \\ \chi_z & \longmapsto & \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\chi_z \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = z^{-\nu_F(a)} = \|a\|_F^{\log_q(z)}$$

- lift χ_z to a character of B trivial on U

- normalized parabolic induction:

$$I_B^G(\chi_z) = \{ f : G \to \mathbb{C} \mid f \text{ locally constant}, \; f(bg) = \chi_z(b)\chi_q(b) f(g) \;\; \forall b \in B, g \in G \}$$

this is a smooth $SL_2(\mathbb{F})$-rep via

$$(h \cdot f)(g) = f(g \cdot h) \qquad g, h \in SL_2(\mathbb{F}).$$

The irreducible subquotients of $I_B^G(\chi)$ with $\chi \in X_{nr}(T)$ form the unramified principal series of $SL_2(\mathbb{F})$.

- $I_B^G(\chi_z) \cong I_B^G(\chi_{z^{-1}})$ irreducible $\quad \forall z \in \mathbb{C}^\times \setminus \{q, q^{-1}, -1\}$
- $I_B^G(\chi_{-1}) = \pi_{-1,s} \oplus \pi_{-1,t}$ direct sum of 2 irreps
- $0 \to triv_G \longrightarrow I_B^G(\chi_{q^{-1}}) \to St_G \longrightarrow 0$
  $0 \to St_G \longrightarrow I_B^G(\chi_q) \to triv_G \to 0$

Unramified L-packets for $SL_2(\mathbb{F})$:

	φ	$\Pi_\varphi(G) = \{\pi \mid \varphi_\pi \cong \varphi\}$
$z \in \mathbb{C}^\times \setminus \{q, q^{-1}, -1\}$	$s = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$, $N = 1$	$I_B^G(\chi_z)$
	$s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $N = 1$	$\{\pi_{-1,t}, \pi_{-1,s}\}$
	$s = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	St_G
	$s = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, $N = 1$	$triv_G$

8. Important subgroups of reductive groups

Def. an algebraic subgroup $P \subset G$ is called parabolic if G/P is a projective variety

- $M \subset P$ Levi subgroup
- $U \subset P$ unipotent radical

Examples in $G = GL_3(\mathbb{F})$

$P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$, $M_0 = T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$, $U_0 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$, $M_1 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$, $U_1 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$P_3 = GL_3(\mathbb{F})$, $M_2 = GL_3(\mathbb{F})$, $U_2 = \{1\}$

- $M \cong P/U$ is also a reductive group
- assume that M contains a maximal $\mathbb{F}$-split torus T
  - $Z(M) \subset T$
  - the derived group M_{der} has finite center
  - $M = M_{der} Z(M) = M_{der} T$

Examples:

$Z(M_0) = M_0$, $M_{0,der} = 1$

$Z(M_1) = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} : a, b \in \mathbb{F}^\times \right\}$, $M_{1,der} = \left\{ \begin{pmatrix} A & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 \end{smallmatrix} & 1 \end{pmatrix} : A \in SL_2(\mathbb{F}) \right\} \cong SL_2(\mathbb{F})$

$Z(M_2) = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix} : a \in \mathbb{F}^\times \right\} = Z(GL_3(\mathbb{F}))$, $GL_3(\mathbb{F})_{der} = SL_3(\mathbb{F})$

9. Square-integrable representations

Def. a complex representation (π, V) of G is square-integrable if $\forall v \in V,\ \lambda \in V^*$, the function $G \to \mathbb{C} : g \mapsto \lambda(\pi_g(v))$ lies in $L^2(G)$.

- Only possible if $Z(G)$ is compact.
- In that case every irr. supercuspidal G-representation is square-integrable
- every irr. smooth square int. G-rep. can be completed to a unitary G-rep on a Hilbert space

Def. (π, V) is essentially square integrable if $(\mathrm{Res}^G_{G_{der}} \pi, V)$ is square int.

- $\omega \in \mathrm{Irr}(M)$ essentially square int.
- $\chi_\omega : Z(M) \to \mathbb{C}^\times$ central character
- $\log|\chi_\omega| : Z(M) \to \mathbb{R}$ vanishes on $Z(M)_{cpt}$ because $\mathbb{R}$ has no compact subgroups except $\{0\}$
- $\log|\chi_\omega| \in \mathrm{Hom}(Z(M)/Z(M)_{cpt}, \mathbb{R}) \cong X^*(Z(M)) \otimes_{\mathbb{Z}} \mathbb{R}$ as $Z(M)/Z(M)_{cpt} \cong X_*(Z(M))$
- P determines a closed "positive cone" $(X^*(Z(M)) \otimes \mathbb{R})^+$

Example: $P_0 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \subset GL_3(F)$, $T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} = M_0 = Z(M_0)$

P_0 yields coroots $\check{\alpha} : x \mapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix}$, $\check{\beta} : x \mapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix}$

$X^*(T) \otimes \mathbb{R}$ has basis $\{\chi_1, \chi_2, \chi_3\}$ with $\chi_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i$

$$(X^*(T) \otimes \mathbb{R})^+ := \{\chi \in X^*(T) \otimes \mathbb{R} \mid \langle \check{\alpha}, \chi \rangle \geq 0, \langle \check{\beta}, \chi \rangle \geq 0\}$$
$$= \{\lambda_1 \chi_1 + \lambda_2 \chi_2 + \lambda_3 \chi_3 \mid \lambda_1 \geq \lambda_2 \geq \lambda_3\}$$

10. the extended Langlands classification

Def. an essentially square int. $\omega \in \mathrm{Irr}(M)$ is in Langlands position w.r.t. P if $\chi_\omega \in (X^*(Z(M)) \otimes_{\mathbb{Z}} \mathbb{R})^+$

Theorem (Langlands, Solleveld)
Let $\pi \in \mathrm{Irr}(G)$, G reductive p-adic group
(a) There exist P, M, ω as above, such that π is isomorphic with a quotient of $I_P^G(\omega)$
normalized parabolic induction
(b) (M, ω) is unique up to conjugation in G.

- in general $I_P^G(\omega)$ has several inequivalent irreducible quotients
- relation with the local Langlands correspondence:
  $\varphi_\pi = \varphi_\omega$ via $M^\vee \hookrightarrow G^\vee$ (conjectural)
- this reduces the LLC to essentially square-integrable representations (of Levi subgroups)

The ABPS conjecture

1. Extended quotients

- W : finite group acting on a space X
- $\tilde{X} := \{ (w,x) \in W \times X \mid w \cdot x = x \}$
- $\tilde{X}_2 = \{ (x,\rho) \mid x \in X, \ \rho \in \mathrm{Irr}(W_x) \}$

  $W_x = \{ w \in W : w \cdot x = x \}$

- W-actions on $\tilde{X}$ and $\tilde{X}_2$:

  $g \cdot (w,x) = (gwg^{-1}, g \cdot x)$

  $g \cdot (x,\rho) = (g \cdot x, g \cdot \rho)$

  $g \cdot \rho : w \mapsto \rho(g^{-1}wg)$

~~Example:~~

Def. the extended quotient of the first kind is

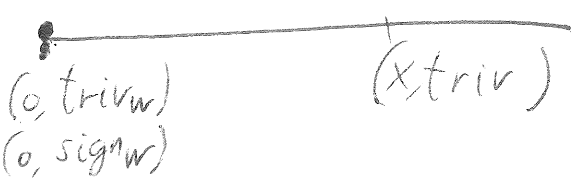
$$X /\!/ W := \tilde{X} / W$$

" second kind is

$$(X /\!/ W)_2 := \tilde{X}_2 / W$$

Example: $X = \mathbb{R}$, $W = \{1,-1\}$

$X /\!/ W = \{1\} \times [0,\infty) \cup \{-1\} \times \{0\}$

$(X /\!/ W)_2 =$ 

- $X/W \cong \bigsqcup_{w \in W/\text{conjugation}} \{w\} \times X^w / Z_W(w)$

  $Z_W(w)$: centralizer of w in W

  $X^w = \{ x \in X : w \cdot x = x \}$,

2. Additional structure of extended quotients

Suppose that X is an affine algebraic variety$/\mathbb{C}$ and that W acts on X by algebraic automorphisms.

- $X/\!/W$ is a disjoint union of affine varieties
- $(X/\!/W)_2$ is a non-separated variety

- $\mathcal{O}(X) = \{$regular functions $X \to \mathbb{C}\}$
- $\mathcal{O}(X) \rtimes W = \mathcal{O}(X) \otimes \mathbb{C}[W]$ with $\underset{\mathcal{O}(X)}{\underset{\cup}{w \cdot f}} \cdot w^{-1} = \underbrace{(x \mapsto f(w^{-1}x w))}_{\mathcal{O}(X)}$

Theorem (Mackey, Clifford)

$\exists$ natural bijection

$$(X/\!/W)_2 \longleftrightarrow \mathrm{Irr}(\mathcal{O}(X) \rtimes W)$$

$$(x, \underset{\mathrm{Irr}(W_x)}{\underset{\cup}{\rho}}) \longleftrightarrow \mathrm{Ind}_{\mathcal{O}(X) \rtimes W_x}^{\mathcal{O}(X) \rtimes W} (\mathbb{C}_x \otimes \rho)$$

Example: $G = SL_3(\mathbb{F})$, $M =$ diagonal torus

ζ: ramified character of $\mathbb{F}^\times$ of order 3 $\quad (3 \nmid q)$

$\sigma = \begin{pmatrix} 1 & & \\ & \zeta & \\ & & \zeta^2 \end{pmatrix} \in \mathrm{Irr}_{cusp}(M)$, $\quad \mathfrak{s} = [M, \sigma]_G$

$T_\mathfrak{s} \cong X_{nr}(M) \cong (\mathbb{C}^\times)^3/\mathrm{diag}$ $\quad W_\mathfrak{s} = \left\{1, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}$

$T_\mathfrak{s}/\!/W_\mathfrak{s} \cong T_\mathfrak{s}/W_\mathfrak{s} \sqcup 6$ points

$(T_\mathfrak{s}/\!/W_\mathfrak{s})_2 \cong \mathrm{Irr}^\mathfrak{s}(G)$, coming from

$$\mathrm{Rep}^\mathfrak{s}(G) \cong \mathrm{Mod}(\mathcal{O}(T_\mathfrak{s}) \rtimes W_\mathfrak{s})$$

$$V \longmapsto \sigma\text{-isotypical component of } V^{\underset{\text{Iwahori group}}{\underset{\uparrow}{I}}}$$

3. Rough form of the conjecture

- $\mathrm{Irr}_{cusp}(G)$: collection of irreducible supercuspidal G-representations, up to equivalence
- $\mathcal{L}evi(G)$: set of representatives for the Levi subgroups of G up to conjugation
- $W(G,M) = N_G(M)/M$: "Weyl group" of $M \in \mathcal{L}evi(G)$
- $\mathfrak{B}(G)$: set of inertial equivalence classes for G

for $\mathfrak{s} = [M, \sigma]_G \in \mathfrak{B}(G)$:

- $T_\mathfrak{s} = \{ \sigma \otimes \chi \mid \chi \in X_{nr}(M) \} \subset \mathrm{Irr}_{cusp}(M)$
- $W_\mathfrak{s} = \{ w \in W(G,M) \mid w \cdot (\sigma \otimes \chi) \in T_\mathfrak{s} \ \forall \chi \in X_{nr}(M) \}$
- $\mathrm{Irr}^\mathfrak{s}(G) = \{$ irr. subquotients of $I_P^G(\sigma \otimes \chi)$, $\chi \in X_{nr}(M) \}$

Conjecture (ABPS 3, part 1)

Let G be a split reductive group over a local nonarchimedean field. $\exists$ canonical bijection

$$\mathrm{Irr}(G) \longleftrightarrow \bigsqcup_{M \in \mathcal{L}evi(G)} \left(\mathrm{Irr}_{cusp}(M) /\!/ W(G,M) \right)_2 = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \left(T_\mathfrak{s} /\!/ W_\mathfrak{s} \right)_2$$

- provides an easy description of $\mathrm{Irr}(G)$ if $\mathrm{Irr}_{cusp}(M)$ is known for all $M \in \mathcal{L}evi(G)$
- for $\mathrm{Irr}_{cusp}(G)$ it does not say anything new

4 The unramified principal series of $G = SL_2(F)$

- $M =$ diagonal torus in G
- $\mathfrak{s} = [M, \mathrm{triv}_M]_G$
- $W_\mathfrak{s} = W(G, M) = \{1, s_\alpha\}$
- $T_\mathfrak{s} = X_{nr}(M) \cong \check{M} = \{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \in PGL_2(\mathbb{C}) \mid z \in \mathbb{C}^\times \} \cong \mathbb{C}^\times$
- action of $W_\mathfrak{s}$ on $T_\mathfrak{s}$: $s_\alpha(z) = z^{-1}$
- $T_\mathfrak{s} /\!/ W_\mathfrak{s} = \{1\} \times T_\mathfrak{s}/W_\mathfrak{s} \cup \{s_\alpha\} \times \{1, -1\}$

$(T_\mathfrak{s}/\!/W_\mathfrak{s})_2$	$\mathrm{Irr}(G)$	$\Phi(G)$	
$z^{\pm 1}, \rho = \mathrm{triv}$	$I_B^G(\chi_z)$	$z^{\pm 1}, N = 1$	z generic
$q^{\pm 1}, \rho = \mathrm{triv}$	triv_G	$q^{\pm 1}, N = 1$	
$-1, \rho = \mathrm{triv}_{W_\mathfrak{s}}$	$\pi_{-1,t}$	$-1, N = 1$	
$-1, \rho = \mathrm{sign}_{W_\mathfrak{s}}$	$\pi_{-1,s}$	$-1, N = 1$	
$1, \rho = \mathrm{triv}_{W_\mathfrak{s}}$	$I_B^G(\chi_1)$	$1, N = 1$	
$1, \rho = \mathrm{sign}_{W_\mathfrak{s}}$	St_G	$q, N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	

- canonical for $z = -1$ by the use of R-groups
- canonical for $z = 1$ by the Springer correspondence for $W_\mathfrak{s}$

5. Infinitesimal central characters

Def. the space of inf. central characters for $\mathfrak{s} = [M,\sigma]_G$
is $T_\mathfrak{s}/W_\mathfrak{s} = \{\sigma\otimes\chi \mid \chi\in X_{nr}(M)\}/W_\mathfrak{s}$
$\pi\in Irr^\mathfrak{s}(G)$ has inf. cc. $W_\mathfrak{s}\cdot\sigma\otimes\chi$ iff
π is a subquotient of $I_P^G(\sigma\otimes\chi)$

Problem: determine the irreducible subquotients of $I_P^G(\sigma\otimes\chi)$

Observation:

$$\begin{tikzcd} Irr^\mathfrak{s}(G) \arrow[r, leftrightarrow] \arrow[dr, "inf. cc."'] & (T_\mathfrak{s}//W_\mathfrak{s})_2 \arrow[d, "pr"] \\ & T_\mathfrak{s}/W_\mathfrak{s} \end{tikzcd}$$

is not far from commutative

Conjecture (ABPS part 2)
There exist:
- a bijection $\varepsilon : T_\mathfrak{s}//W_\mathfrak{s} \to (T_\mathfrak{s}//W_\mathfrak{s})_2$ preserving the $T_\mathfrak{s}$-coordinates
- for every component c of $\widetilde{T_\mathfrak{s}}$ a "correcting cocharacter" $h_c : \mathbb{C}^\times \to T_\mathfrak{s}$
- a map $\theta_z : T_\mathfrak{s}//W_\mathfrak{s} \to T_\mathfrak{s}/W_\mathfrak{s}$ for $z\in\mathbb{C}^\times$
  $c \ni (w,t) \longmapsto h_c(z)t/W_\mathfrak{s}$

such that:

$$\begin{tikzcd} Irr^\mathfrak{s}(G) \arrow[ddr, "inf. cc."'] & & (T_\mathfrak{s}//W_\mathfrak{s})_2 \arrow[ll, "\mu_\mathfrak{s}"'] \arrow[ddl, bend left, "pr"] \\ & T_\mathfrak{s}//W_\mathfrak{s} \arrow[ur, "\varepsilon"] \arrow[d, "\theta_z"] & \\ & T_\mathfrak{s}/W_\mathfrak{s} & \end{tikzcd}$$

- $pr\circ\varepsilon = \theta_1$
- $inf.cc.\circ\mu_\mathfrak{s}\circ\varepsilon = \theta_{q^{1/2}}$

Example: $G = SL_2(\mathbb{F})$, $\mathfrak{s} = [T, triv_T]_G$

$\widetilde{T_\mathfrak{s}} = \underbrace{\{1\}\times T_\mathfrak{s}}_{h_c = 1} \cup \underbrace{\{s_\alpha\}\times\{-1\}}_{h_c = 1} \cup \underbrace{\{s_\alpha\}\times\{1\}}_{h_c(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1\end{pmatrix},\ W_\mathfrak{s} h_c(q^{1/2}) = inf.cc.(St_G)}$

6. Relation between the Langlands classification and ABPS

- $Q = PL$ parabolic subgroup with Levi factor L
- $M \subset L \subset G$
- $\mathfrak{s} = [M,\sigma]_G$, $\mathfrak{s}^L = [M,\sigma]_L$
- $\omega \in \mathrm{Irr}^{\mathfrak{s}^L}(L)$ unitary and ess. square int.

Then $\forall \chi \in X_{nr}(L)$: $I_Q^G(\omega\otimes\chi) \in \mathrm{Rep}^{\mathfrak{s}}(G)$

The collection of ~~irreducible~~ quotients of $I_Q^G(\omega\otimes\chi)$ with $\cancel{\omega\otimes}\chi \in X_{nr}(L)$ ~~in~~ Langlands ~~position~~ forms a series in $\mathrm{Irr}^{\mathfrak{s}}(G)$. It should be in bijection with a union of components of $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$.

- $\pi \in \mathrm{Irr}(G)$: quotient of $I_Q^G(\omega\otimes\chi)$

  we want to compare the ABPS-parameters of π and $\omega\otimes\chi$

- write $\mathrm{inf.cc.}(\omega) = \sigma_\omega \in T_{\mathfrak{s}}/W_{\mathfrak{s}}^L$, then

$\mathrm{inf.cc}(\pi) = \mathrm{inf.cc.}(\omega\otimes\chi)/W_{\mathfrak{s}} \; \cancel{=} \; W_{\mathfrak{s}}\sigma_\omega \otimes \underset{\in X_{nr}(M)}{\chi|_M}$

- the ABPS-parameter of $\omega\otimes\chi$ should be (t,ρ^L) with $t = \sigma_\omega|\sigma_\omega|^{-1}\otimes\chi|_M$ and $\rho \in \mathrm{Irr}((W_{\mathfrak{s}}^L)_t)$
- the ABPS-parameter of π should be (t,ρ) with $\rho \in \mathrm{Irr}((W_{\mathfrak{s}})_t)$ such that $\rho|_{(W_{\mathfrak{s}}^L)_t}$ contains ρ^L.
- the correcting cocharacter h_c for (t,ρ) and (t,ρ^L) has $h_c(q^{1/2}) = |\sigma_\omega|$ (well-defined because ω unitary)
- ~~ρ should be determined by φ_π and some Springer-like correspondence for $(W_{\mathfrak{s}})_t$~~

7. Theorem (Harish-Chandra)

- $P = MU$ parabolic subgroup of G
- π M-rep. of finite length
- $w \in W(G,M) = N_G(M)/M$

Then $I_P^G(\pi)$ and $I_P^G(w\cdot\pi)$ have the same irreducible subquotients, with the same multiplicities

Idea of proof

- $I_P^G(w\cdot\pi) \stackrel{\sim}{=} w^{-1} I_P^G(w\cdot\pi) = I_{P'}^G(\pi)$, $P' = w^{-1}Pw$
- for $\chi \in X_{nr}(M)$ with $|\chi|$ "large" construct
  $I(\pi\otimes\chi) : I_P^G(\pi\otimes\chi) \xrightarrow{\sim} I_{P'}^G(\pi\otimes\chi)$
- $tr(I_P^G(\pi\otimes\chi)) = tr(I_{P'}^G(\pi\otimes\chi))$ for many χ

both depend algebraically on χ, so holds $\forall \chi \in X_{nr}(M)$

- $I_P^G(\pi)$ and $I_{P'}^G(\pi)$ have the same trace
- trace(ρ) determines $JH(\rho)$ and multiplicities, for ρ admissible

---

- shows that, for $\rho \in Irr(G)$
  $\text{inf. cc.}(\rho) = (M,\sigma) \Longleftrightarrow \rho$ is a subquotient of $I_P^G(w\cdot\sigma)$ for any $w \in N_G(M)/M$

- the theory of the Bernstein center implies:
  $JH(I_P^G(\pi)) \cap JH(I_P^G(\pi')) = \emptyset$ if $\pi, \pi' \in Irr(M)$ are not $W(G,M)$-associate

8. More difficult example of ABPS

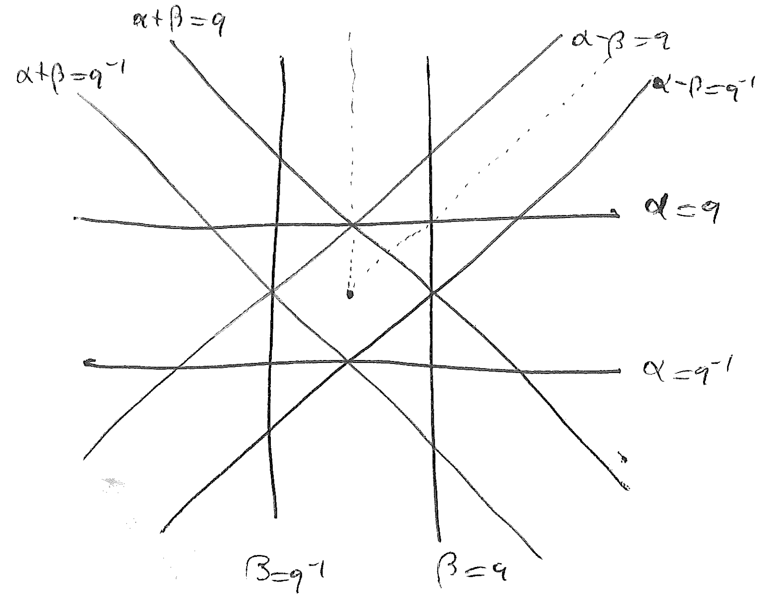
$G = Sp_4(F)$, T split maximal torus in G

coroots: $R^\vee(G,T) = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha-\beta)\}$ (type B_2)

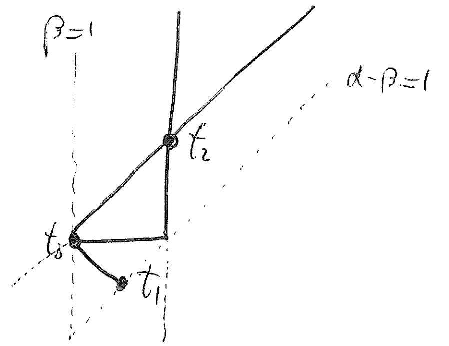
$s = [T, triv_T]_G$, $W_s = W(G,T) \cong D_4$

We look at the family of reps. $I_B^G(\chi)$ with $\chi: T \to \mathbb{R}_{>0}$ unramified

Space of such χ: $\check{T}_{\mathbb{R}} \cong (\mathbb{R}_{>0})^2$



choose a fundamental domain $\check{T}_{\mathbb{R}}^+$ for the action of W_s, ~~because~~ ~~$JH(I_B^G(w\chi)) = JH(I_B^G(\chi))$~~



List of G-irreps in here:

- for all $t \in \check{T}_{\mathbb{R}}^+$: $L(I_B^G(t))$ = K-spherical G-rep with parameter $t \in \check{T}/W$

  Langlands quotient

- on the lines: $I_P^G(St_M \otimes t|_{Z(M)})$ for a suitable parabolic P with $B \subsetneq P \subsetneq G$
- in t_1: $L(I_B^G(t_1))$ and 2 summands of $I_{P_{\alpha+\beta}}^G(St_{M_{\alpha+\beta}} \otimes 1)$
- in t_2: $L(I_B^G(t_2))$, St_G, $L(I_{P_\beta}^G(St_{M_\beta} \otimes q))$, $L(I_{P_{\alpha+\beta}}^G(St_{M_{\alpha+\beta}} \otimes q^3))$

  M_γ: Levi with coroots $\{\pm\gamma\}$

- in t_3: $L(I_B^G(t_3))$, $I_{P_\alpha}^G(St_{M_\alpha} \otimes 1)$, $I_{P_{\alpha-\beta}}^G(St_{M_{\alpha-\beta}} \otimes q)$

9. Relation between the LLC and ABPS

$$\begin{tikzcd} \bigsqcup_{s\in\mathfrak{B}(G)} (T_s/\!/W_s)_2 \arrow[r] \arrow[rr, dashed, bend right] & \mathrm{Irr}(G) \arrow[r] & \Phi(G) \end{tikzcd}$$

Procedure to compute an unramified L-parameter from an ABPS-parameter (t,ρ)

- $\mathfrak{s} = [T,\sigma]_G$, $t \in T_{\mathfrak{s}} \subset \mathrm{Irr}(T)$, $\rho \in \mathrm{Irr}(W_{\mathfrak{s},t})$
- find $\varphi_t : W_F \to T^\vee$ with the LLC for T.
- $W_{\mathfrak{s}t} = W(L^\vee, T^\vee) \rtimes R_t$ for some Levi subgroup $L^\vee \subset G^\vee$
- via the Springer correspondence ρ determines a unipotent element $N_\rho \in L^\vee$, unique up to conjugation by $L^\vee \rtimes R_t \subset G^\vee$, ~~$N_\rho$ commutes with $\varphi_t(W_F)$~~
- N_ρ commutes with $\varphi_t(W_F)$ and $h_c(z) N_\rho h_c(z)^{-1} = N_\rho^{z^2}$

  correcting cocharacter for (w,t)

- $\varphi_{t\rho} : WD_F \to G^\vee$, $(i\,\mathrm{Fr}^n, z) \longmapsto \varphi_t(i\,\mathrm{Fr}^n)\, h_c(q^{n/2})\, N_\rho^z$

  $i \in I_F$, $z \in \mathbb{C}$

Hope: can be done for $[M,\sigma]_G$ if φ_σ is known

Conjecture (ABPS, part 3)

$\exists$ map $\lambda : T_s/\!/W_s \to \{$unipotent classes in ~~Levi~~ subgroups of $G^\vee\}$ such that, for $(w,t), (w',t') \in T_s/\!/W_s$

$(\mu_s \circ \varepsilon)(w,t)$ and $(\mu_s \circ \varepsilon)(w',t')$ are in the same L-packet

$$\Updownarrow$$

$$\lambda(w,t) = \lambda(w',t') \text{ and } \underbrace{\theta_z(w,t)}_{= h_c(z)t/W_s} = \theta_z(w',t') \quad \forall z \in \mathbb{C}^\times$$

10. L-packets

Def. $\Pi_\varphi(G) = \{\pi \in \mathrm{Irr}(G) :$ the L-parameter of π is $\check{G}$-conjugate to $\varphi\}$

Conjectural properties:

- finite
- if one element of $\Pi_\varphi(G)$ is ess. square integrable (~~or tempered~~) then so are all elements
- same for tempered
- an L-packet can be distributed over various Bernstein components
- Suppose that $\omega_1, \omega_2 \in \Pi_\varphi(M)$ are ess. square integrable and that π_i is a Langlands quotient of $I_P^G(\omega_i)$. Then π_1 and π_2 belong to the same L-packet. This should determine most L-packets.
- if G is split, $\Pi_\varphi(G)$ is in bijection with $\mathrm{Irr}(\underbrace{\pi_0(Z_{\check{G}}(\varphi))}_{\text{component group of } \varphi})$
- Example: for the reps. of $Sp_4(F)$ above, the only non-singleton L-packet is formed by the 2 summands of $I_{P_{\alpha+\beta}}^G(St_{M_{\alpha+\beta}} \otimes 1)$

Conjecture (ABPS, part 3)

Fix $\mathfrak{s} = [M, \sigma]_G$. $\exists$ locally constant map

$\lambda : T_\mathfrak{s} /\!/ W_\mathfrak{s} \longrightarrow \{$unipotent classes in subgroups of $\check{G}\}$

For $(w,t), (w',t') \in T_\mathfrak{s} /\!/ W_\mathfrak{s}$

the irreps corresponding to (w,t) and (w',t') are in the same L-packets

$$\Updownarrow$$

$\theta_z(w,t) = \theta_z(w',t')$ $\forall z \in \mathbb{C}^\times$ and $\lambda(w,t) = \lambda(w',t')$

$\quad ||$

$h_c(z)t/W_\mathfrak{s}$

11. Principal series representations

Def. the principal series of G consists of all (irreducible) subquotients of $I_B^G(\chi)$, with B a Borel subgroup and $\chi \in \mathrm{Irr}(T)$ maximal split torus

Theorem. The ABPS conjectures hold for every Bernstein component in the principal series of a split reductive p-adic group (with mild restriction on the residual characteristic of $\mathbb{F}$).

- $\mathfrak{s} = [T, \sigma]_G$, $\varphi_\sigma : WD_F \to \check{T}$, $\check{H} := Z_{\check{G}}(\varphi_\sigma(I_F))$

Proposition

$$W_\mathfrak{s} \cong N_{\check{H}}(\check{T})/\check{T} \cong W(\check{H}^\circ, \check{T}) \rtimes \pi_0(\check{H})$$

Example $G = SL_n(\mathbb{F})$

$\sigma_0 \in \mathrm{Irr}(\mathbb{F}^\times)$, $\zeta \in \mathrm{Irr}(\mathbb{F}^\times / 1+\mathfrak{p}_F)$, $\zeta^d = 1$, $d \mid n$

$$\sigma = \begin{pmatrix} \sigma_1 \cdots \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} = \bigotimes_{i=1}^{d} \sigma_i^{\otimes n/d} = \bigotimes_{i=1}^{d} (\sigma_0 \otimes \zeta^i)^{\otimes n/d}$$

- $\check{H}^\circ \cong P(GL_{n/d}(\mathbb{C})^d)$

  $\pi_0(\check{H}) \cong \mathbb{Z}/d\mathbb{Z}$, generated by $\begin{pmatrix} 0 & & & I_{n/d} \\ I_{n/d} & 0 & & \\ & I_{n/d} & \ddots & \\ & & I_{n/d} & 0 \end{pmatrix}$

- $W_\mathfrak{s} \cong S_n^d \rtimes \mathbb{Z}/d\mathbb{Z}$, because $\sigma \otimes \zeta_{\circ\det} = \sigma$

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write $R_\mathfrak{s} = R(\check{H}^\circ, \check{T})$, then $W_\mathfrak{s} \cong W(R_\mathfrak{s}) \rtimes \pi_0(\check{H})$

- $\pi_0(\check{H}) = \{ w \in W_\mathfrak{s} \mid w(R_\mathfrak{s}^+) = R_\mathfrak{s}^+ \}$
- works because $W(R_\mathfrak{s})$ acts simply transitively on the set of positive systems in $R_\mathfrak{s}^+$

§12. Hecke algebras for the principal series

- $\check{T}, R_s, q$ determine an affine Hecke algebra $\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), q)$

rough structure of AHAs:

- $\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), q) \cong \mathbb{C}[X^*(\check{T}) \rtimes W(R_s)] \cong \mathcal{O}(T_s) \otimes \mathbb{C}[W(R_s)]$ as vector spaces
- $\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), 1) = \mathbb{C}[X^*(\check{T}) \rtimes W(R_s)] \cong \mathcal{O}(T_s) \rtimes W(R_s)$
- with respect to suitable bases, the multiplication in $\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), q)$ depends polynomially on $q^{1/2} \in \mathbb{C}^\times$
- q can be regarded as a parameter in $\mathbb{C}^\times$

Lusztig developed a theory which provides canonical bijections
$$\mathrm{Irr}(\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), q)) \longleftrightarrow \mathrm{Irr}(\mathbb{C}[X^*(\check{T}) \rtimes W(R_s)]) \cong (\check{T} /\!/ W(R_s))_2$$

Theorem (Roche)
$\mathrm{Rep}^s(G)$ is equivalent with $\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), q) \rtimes \pi_0(\check{H})$, with respect to the actions of $\pi_0(\check{H}) \subset W_s$ on $T_s \cong \check{T}$ and on $W(R_s) \triangleleft W_s$.

Consequence: there are canonical bijections

$$\begin{tikzcd}
\mathrm{Irr}^s(G) \arrow[rr, leftrightarrow, dashed] \arrow[d, leftrightarrow] & & (T_s /\!/ W_s)_2 \arrow[d, leftrightarrow] \\
\mathrm{Irr}(\mathcal{H}(X^*(\check{T}) \rtimes W(R_s), q) \rtimes \pi_0(\check{H})) \arrow[rr, leftrightarrow] & & \mathrm{Irr}(\mathcal{O}(T_s) \rtimes W(R_s) \rtimes \pi_0(\check{H}))
\end{tikzcd}$$