

L-Groups

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$$GL_n \longleftrightarrow GL_n$$

$$SL_n \longleftrightarrow PGL_n$$

$$SO_{2n} \longleftrightarrow SO_{2n}$$

$$Sp_{2n} \longleftrightarrow SO_{2n+1}$$

- 1 Motivation
- 2 Reductive Groups and Root Datum
- 3 The L-Group
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F - local non-archimedean field

$\mathcal{O} = \mathcal{O}_F$ - ring of integers

G - connected split reductive group / F

B - Borel subgroup of G

T - maximal split torus

W - Weyl group

Motivation

The **spherical Hecke algebra** is defined as

$$\mathcal{H} = C_c^\infty (K \backslash G(F) / K)$$

together with the multiplication

$$f_1 * f_2 (g) = \int_{G(F)} f_1 (gh^{-1}) f_2 (h) dh$$

Fact

This is a unital algebra with $1_{\mathcal{H}} = e_K$.

Theorem

\mathcal{H} is a commutative algebra.

We then have

Theorem (Satake Isomorphism - Version I)

$$\mathcal{H} \cong \mathbb{C}[X_*(T)]^W$$

Let $\mathbb{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$, then $X^*(\mathbb{T}) \cong X_*(T)$ and $X_*(\mathbb{T}) \cong X^*(T)$.
If $G = T$, the theorem yields

$$\mathcal{H} \cong \text{Rep}(\mathbb{T}).$$

Goal

There exists a complex reductive group ${}^L G$ such that

$$\mathcal{H} \cong \text{Rep}({}^L G)$$

and such that \mathbb{T} is a maximal torus of ${}^L G$.

Root Datum

A **root datum** is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, where

- ① X, X^\vee - free \mathbb{Z} -modules of finite rank
- ② perfect pairing $X \times X^\vee \rightarrow \mathbb{Z}$ denoted by $\langle \cdot, \cdot \rangle$,
- ③ $\Phi \subset X, \Phi^\vee \subset X^\vee$ - finite subsets and
- ④ there is a bijection

$$\begin{aligned} \Phi &\rightarrow \Phi^\vee \\ \alpha &\mapsto \alpha^\vee, \end{aligned}$$

satisfying $\forall \alpha \in \Phi$

RD1 $\langle \alpha, \alpha^\vee \rangle = 2$

RD2 $s_\alpha \Phi = \Phi, s_{\alpha^\vee} \Phi^\vee = \Phi^\vee$

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

$$s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

Root Datum Attached to a Reductive Group

$$G \rightsquigarrow \Psi = \Psi(G) = \Psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$$

$$X = X^*(T) = \text{Hom}(T, \mathbb{G}_m)$$

$$X^\vee = X_*(T) = \text{Hom}(\mathbb{G}_m, T).$$

$$\chi \in X, \phi \in X^\vee \quad \chi \circ \phi \in \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$

We set

$$\langle \chi, \phi \rangle = \chi \circ \phi \in \mathbb{Z}$$

$$\chi(\phi(t)) = t^{\langle \chi, \phi \rangle}$$

$\mathfrak{g} = \text{Lie}(G) = \mathfrak{h} \oplus \bigoplus_{\alpha \in X} \mathfrak{g}_\alpha$, where $\mathfrak{h} = \text{Lie}(T)$ and T acts on \mathfrak{g}_α by α .

$$\Phi = \{\alpha \mid \mathfrak{g}_\alpha \neq \{0\}\}$$

$\alpha \rightsquigarrow s_\alpha$

$$\exists! \alpha^\vee \in X^\vee : \quad \forall x \in X : s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

$$\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$$

Existence and Uniqueness Theorem

$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is **reduced** if

$$\forall \alpha \in \Phi, c \in \mathbb{Q} \quad c\alpha \in \Phi \Rightarrow c = \pm 1.$$

Theorem (Springer theorems 9.6.2 and 10.1.1)

There is a bijection

$$\{\text{Split connected reductive groups over } k\} \leftrightarrow \{\text{Reduced root datum}\}$$

taking a group G to $\Psi(G)$.

Dual Root Datum and the L-group

From $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ we form the **dual root datum**

$$\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi)$$

\rightsquigarrow There is a unique G^\vee defined over \mathbb{C} - **the dual group**

$$(\Psi^\vee)^\vee = \Psi$$

Example I - $G = GL_2$

$$T = \left\{ \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mid t_1, t_2 \in k^\times \right) \right\}.$$

$$\chi_1 \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right) = t_1, \quad \chi_2 \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right) = t_2$$

$$\phi_1(t_1) = \begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}, \quad \phi_2(t_2) = \begin{pmatrix} 1 & \\ & t_2 \end{pmatrix}.$$

$$X = X^*(T) = \{ \chi_1^n \chi_2^m \mid m, n \in \mathbb{Z} \}$$

$$X^\vee = X_*(T) = \{ \phi_1^n \phi_2^m \mid m, n \in \mathbb{Z} \}$$

$$\langle \chi_1, \phi_1 \rangle = 1, \quad \langle \chi_1, \phi_2 \rangle = 0, \quad \langle \chi_2, \phi_1 \rangle = 0, \quad \langle \chi_2, \phi_2 \rangle = 1$$

$$\Phi = \{ \alpha, -\alpha \}, \quad \alpha = \chi_1 \chi_2^{-1}, \quad \Phi^\vee = \{ \alpha^\vee, -\alpha^\vee \}, \quad \alpha^\vee = \phi_1 \phi_2^{-1}$$

This root datum is obviously self dual and hence $GL_2^\vee = GL_2$.

Maps of Root Data

Given $\psi_1 = (X_1, \Phi_1, X_1^\vee, \Phi_1^\vee)$ and $\psi_2 = (X_2, \Phi_2, X_2^\vee, \Phi_2^\vee)$ a map $\psi_1 \rightarrow \psi_2$ is

$$f : X_2 \rightarrow X_1$$

such that

- 1 $f|_{\Phi_2}$ is a bijection onto Φ_1
- 2 $\forall \alpha \in \Phi_2 \ f^\vee(f(\alpha)^\vee) = \alpha^\vee$

If f is a bijection, we say that ψ_1 and ψ_2 are isomorphic. This is equivalent to isomorphism in the level of groups.

Example II - $G = SL_2$

$$T = \left\{ \left(\begin{array}{c|c} t & \\ \hline & t^{-1} \end{array} \right) \mid t \in k^\times \right\}$$

$$X = X^*(T) = \{ \chi_1^n \mid n \in \mathbb{Z} \}$$

$$X^\vee(T) = \{ \phi^n \mid n \in \mathbb{Z} \}, \quad \phi = \phi_1 \phi_2^{-1}$$

$$\langle \chi_1, \phi \rangle = 1$$

$$\Phi = \{ \alpha, -\alpha \}, \quad \alpha = \chi_1^2$$

$$\Phi^\vee = \{ \alpha^\vee, -\alpha^\vee \}, \quad \alpha^\vee = \phi.$$

This root datum is obviously not self dual since

$$\mathbb{Z} \cdot \Phi \subsetneq X, \quad \mathbb{Z} \cdot \Phi^\vee = X^\vee.$$

How can we find SL_2^\vee ?

- 1 We can use the proof of the existence theorem (which is constructive) and construct SL_2^\vee as a Chevalley group.
- 2 If one knows a list of algebraic groups and their root datum he can try and recover the dual group from the list.

We pursue the second option.

Root Datum of PGL_2

$$R = \left\{ \left(\begin{array}{c|c} t & \\ \hline & 1 \end{array} \right) \middle| t \in k^\times \right\}$$

$$Y = X^*(R) = \{ \chi_1^n \mid n \in \mathbb{Z} \}$$

$$Y^\vee = X_*(R) = \{ \phi_1^n \mid n \in \mathbb{Z} \}$$

$$\Xi = \{ \beta, -\beta \}, \quad \beta = \chi_1$$

$$\Xi^\vee = \{ \beta^\vee, -\beta^\vee \}, \quad \beta^\vee = \phi_1^2$$

Let

$$f : Y \rightarrow X^\vee$$

$$\chi_1 \mapsto \phi = \phi_1 \phi_2^{-1}$$

Then

$$f^\vee : X \rightarrow Y^\vee$$

$$\chi_1 \mapsto \phi_1$$

and in particular

$$f(\beta) = \alpha^\vee, f(\beta)^\vee = \alpha.$$

Hence

$$(X^\vee, \phi^\vee, X, \phi) \cong (Y, \Xi, Y^\vee, \Xi^\vee),$$

and

$$SL_2^\vee = PGL_2.$$

L-Groups of Classical Groups

$$GL_n \longleftrightarrow GL_n \quad (A_n)$$

$$SL_n \longleftrightarrow PGL_n \quad (A_n)$$

$$Sp_{2n} \longleftrightarrow SO_{2n+1} \quad (B_n \leftrightarrow C_n)$$

$$SO_{2n} \longleftrightarrow SO_{2n} \quad (D_n)$$

$$Spin_{2n+1} \longleftrightarrow Sp_{2n}/Z \quad (B_n \leftrightarrow C_n)$$

$$Spin_{2n} \longleftrightarrow SO_{2n}/Z \quad (D_n)$$

$$\text{Adjoint} \longleftrightarrow \text{Simply connected}$$

G is called **adjoint** if $X^*(T) = \mathbb{Z}[\Phi]$, example: PGL_2 .

G is called **simply connected** if $X_*(T) = \mathbb{Z}[\Phi^\vee]$, example: SL_2 .

Back to Satake Isomorphism

Theorem (Satake Isomorphism - Version II)

For G split

$$\mathcal{H} \cong \text{Rep}(G^{\vee}(\mathbb{C}))$$

We can also dualize this

Theorem (Satake Isomorphism - Version III)

There is bijection

$$\left\{ \begin{array}{l} \text{unramified} \\ \text{representations} \\ \text{of } G(F) \end{array} \right\} / \sim \leftrightarrow \left\{ \begin{array}{l} \text{semisimple} \\ \text{conjugacy classes} \\ \text{in } G^{\vee}(\mathbb{C}) \end{array} \right\}$$

The Non-split Case

We would like to generalize the Satake isomorphism to the case of a non-split case.

F - local non-archimedean field

\overline{F} its algebraic closure

$\Gamma = \text{Gal}(\overline{F}/F)$

G - reductive group defined over F

Facts

- $G_{\bar{F}}$ is split.
- $\exists \Gamma \triangleleft G_{\bar{F}} : G = G_{\bar{F}}^{\Gamma}$

$$G_{\bar{F}} \rightsquigarrow \Psi(G_{\bar{F}}), G_{\bar{F}}^{\vee}$$

$$\Gamma \triangleleft G_{\bar{F}} \rightsquigarrow \Gamma \triangleleft \Psi(G_{\bar{F}}) \rightsquigarrow \Gamma \triangleleft \Psi(G_{\bar{F}})^{\vee} \rightsquigarrow \Gamma \triangleleft G_{\bar{F}}^{\vee}$$

$${}^L G = G_{\bar{F}}^{\vee} \rtimes \text{Gal}(\bar{F}/F)$$

Theorem (Satake Isomorphism - Version III in General)

There is bijection

$$\left\{ \begin{array}{l} \text{unramified representations} \\ \text{of } G(F) \end{array} \right\} / \sim \leftrightarrow \left\{ \begin{array}{l} \text{semisimple conjugacy} \\ \text{classes in } {}^L G(\mathbb{C}) \end{array} \right\}$$

Example - $U(3)$

E/F - quadratic extension

$$\Gamma = \text{Gal}(E/F) = \{1, \sigma\}$$

On E^3 we define a sesqui-linear form by

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 \sigma(y_1) + x_2 \sigma(y_2) + x_3 \sigma(y_3)$$

$$U(3) = \left\{ g \in \text{GL}_3(E) \mid \forall u, v \in E^3 \langle u, v \rangle = \langle g \cdot u, g \cdot v \rangle \right\},$$

We now compute ${}^L U(3)$.

$$U(3)_E \cong GL_3$$

$$\langle u, v \rangle = \langle g \cdot u, g \cdot v \rangle = \langle u, (\sigma(Tg)g) \cdot v \rangle$$

$$U(3) = GL_3^{\text{Gal}(E/F)}, \quad \sigma \cdot g = \sigma(Tg^{-1})$$

The corresponding action of $\text{Gal}(E/F)$ on $GL_3^{\vee} = GL_3$ is given by

$$\sigma \cdot g = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} Tg^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

We conclude by setting

$${}^L U(3) = GL_3(\mathbb{C}) \rtimes \Gamma$$

Thank You!