Local
Langlands correspondence
and examples
of ABPS
conjecture

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Notation

- F non-archimedean local field : finite extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$
- $O_F = \{x \in F, v(x) \ge 0\}$, ring of integers of F
- $\mathfrak{p}_F = \{x \in F, v(x) > 0\}$, unique maximal ideal of O_F
- $k_F = O_F/\mathfrak{p}_F \simeq \mathbb{F}_q$, residual field of F (finite field)
- $\varpi_F \in O_F$ uniformizer of F
- \overline{F} separable algebraic closure of F

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$$1 \longrightarrow I_{K/F} \longrightarrow Gal(K/F) \longrightarrow Gal(k_K/k_F) \longrightarrow 1$$

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An extension K/F is unramified if the ramification index e(K/F) = 1 and k_K/k_F is separable.

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Theorem

 $\forall n \geqslant 1, \exists ! \ F \subset F_n \subset \overline{F} \ unramified extension of degree n$ F_n/F is Galois, $k_{F_n} \simeq \mathbb{F}_{q^n}$ and $Gal(F_n/F) \stackrel{\sim}{\longrightarrow} Gal(k_{F_n}/k_F)$

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$$\begin{split} \phi_n &\in Gal(F_n/F) \leftrightarrow (x \mapsto x^q) \in Gal(k_{F_n}/k_F) \\ \Phi_n &= \phi_n^{-1}, \ \Phi_n \mapsto 1, \ Gal(F_n/F) \simeq \mathbb{Z}/n\mathbb{Z} \\ F_{ur} &= \text{composite of all } F_n \\ F_{ur}/F \text{ is the unique maximal unramified extension of } F \\ Gal(F_{ur}/F) &\simeq \varprojlim_{n \geqslant 1} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}} \\ \Phi_F &\in Gal(F_{ur}/F), \ \Phi_F|_{F_n} = \Phi_n \end{split}$$

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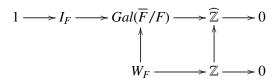
Topology on Γ_F : Open neighborhood basis at the identity are $Gal(\overline{F}/K)$ with K/F finite extension Γ_F is a compact Haussdorf profinite group

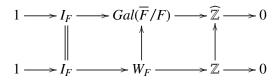
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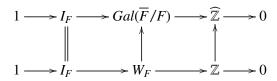
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Definition

The Weil group of F is the topological group, with underlying abstract group the inverse image in Γ_F of $<\Phi>$, such that :

- I_F is an open subgroup of W_F
- the topology on I_F , as subspace of W_F , coincides with its natural topology as subspace of Γ_F

Let $W_F^{der} = \overline{[W_F, W_F]}$ and $W_F^{ab} = W_F / W_F^{der}$.

Artin Reciprocity Map

There is a canonical continuous group morphism

$$a_F:W_F\longrightarrow F^{\times}$$

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$$\left\{\begin{array}{c} \text{irreducible representations of } F^\times \\ GL_1(F) \longrightarrow GL_1(\mathbb{C}) \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \text{admissible morphisms} \\ W_F \longrightarrow GL_1(\mathbb{C}) \end{array}\right\}$$

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If T is a split torus over F, then

$$\left\{\begin{array}{c} \text{irreducible representations of } T \\ T \longrightarrow GL_1(\mathbb{C}) \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \text{admissible morphisms} \\ W_F \longrightarrow \widehat{T} \end{array}\right\}$$

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(+ equality of *L*-functions)

This is the local Langlands correspondence for GL(1) and for split tori.

For $\psi:W_F\longrightarrow GL_n(\mathbb{C})$ we can define a L-function by

$$L(s, \psi) = det(1 - \psi(\Phi)|_{V^{I_F}} q^{-s})^{-1}$$

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Godement and Jacquet defined *L*-functions for the representations of $GL_n(F)$. For n = 2,

- for supercuspidal representations are 1
- for Steinberg representation (and their twist by character) involve one factor with q^{-s}
- for other involve two factors with q^{-s}

Definition

The Weil-Deligne group is $WD_F = \mathbb{C} \rtimes W_F$ where W_F acts on \mathbb{C} by $wxw^{-1} = |w|x$ for $w \in W_F, x \in \mathbb{C}$.

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A Langlands parameter for GL(n) is a continous morphism $\psi:WD_F\longrightarrow GL_n(\mathbb{C})$ such that the restriction to \mathbb{C} is algebraic, the image of \mathbb{C} consists of unipotent elements and the image of W_F consists of semisimple elements.

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We can identify one Langlands parameter ψ to a pair (ρ, N) where $\rho: W_F \longrightarrow GL_n(\mathbb{C})$ is a continous morphism such that his image consists of semisimple elements and $N \in \mathfrak{gl}_n(\mathbb{C})$ is such that $\rho(w)N\rho(w)^{-1} = |w|N$ for all $w \in W_F$.

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$$\psi(z, w) = \exp(zN)\rho(w), (z, w) \in WD_F$$

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This group is isomorphic to $SL_2(\mathbb{C}) \times W_F$ via the map $(x, w) \mapsto (xh_w, w)$.

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Local Langlands correspondence (conjecture)

Let G a split connected reductive group over F and \widehat{G} is Langlands dual. Let $\Pi(G)$ the set of isomorphism classes of irreducible representations of G and $\Phi(G)$ the set of equivalence classes of admissible morphisms for G. There is finite-to-one map $\operatorname{rec}:\Pi(G)\longrightarrow\Phi(G)$ with these properties

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• $\pi \in \Pi(G)$ is essentially square-integrable, if and only if, the image of W_F' by $\phi_\pi:W_F' \longrightarrow \widehat{G}$ is not contained in any proper Levi subgroup of \widehat{G} .

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- $\pi \in \Pi(G)$, χ a character of F^{\times} , then $rec(\pi \otimes \chi) = rec(\pi)\chi$.
- Equality of *L*-functions and ε -factors.

$$\operatorname{Let} J = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix} \text{ and } J' = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

For $M \in GL_4(F)$, ${}^{\tau}M = J^tMJ$. Let

$$G=Sp_4(F)=\{g\in SL_4(F)|^tgJ'g=J'\}$$

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There are three parabolic subgroups of $Sp_4(F)$: B, P, Q with respective Lévi T, M, L.

•
$$T = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix}, \ t_i \in F^{\times} \right\}, \ T \simeq (F^{\times})^2$$

•
$$T = \begin{cases} \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_2^{-1} \\ & & t_1^{-1} \end{pmatrix}, \ t_i \in F^{\times} \end{cases}$$
, $T \simeq (F^{\times})^2$
• $M = \left\{ \begin{pmatrix} g & \\ & {}^{\tau}g^{-1} \end{pmatrix}, \ g \in GL_2(F) \right\}$, $M \simeq GL_2(F)$

$$\bullet \ T = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_1^{-1} \\ & & t_1^{-1} \end{pmatrix}, \ t_i \in F^{\times} \right\}, \ T \simeq (F^{\times})^2$$

$$\bullet \ M = \left\{ \begin{pmatrix} g & & \\ & \tau_g^{-1} \\ & & \end{pmatrix}, \ g \in GL_2(F) \right\}, \ M \simeq GL_2(F)$$

$$\bullet \ L = \left\{ \begin{pmatrix} t & & \\ & g & \\ & & t^{-1} \\ \end{pmatrix}, t \in F^{\times}, g \in SL_2(F) \right\}, \ L \simeq GL_1(F) \times SL_2(F)$$

The Weyl group of G is $W = N_G(T)/T \simeq \mathfrak{S}_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2$, generated by

$$a = \begin{pmatrix} 1 & & & \\ -1 & & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, b = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}$$

Action of W on T and on their irreducible representations

$$\begin{array}{c|cccc} w & w \cdot (t_1, t_2) & w \cdot (\chi_1 \boxtimes \chi_2) \\ e & (t_1, t_2) & \chi_1 \boxtimes \chi_2 \\ a & (t_2, t_1) & \chi_2 \boxtimes \chi_1 \\ b & (t_1, t_2^{-1}) & \chi_1 \boxtimes \chi_2^{-1} \\ ab & (t_2^{-1}, t_1) & \chi_2 \boxtimes \chi_1^{-1} \\ ba & (t_2, t_1^{-1}) & \chi_2^{-1} \boxtimes \chi_1 \\ aba & (t_1^{-1}, t_2) & \chi_1^{-1} \boxtimes \chi_2 \\ bab & (t_2^{-1}, t_1^{-1}) & \chi_2^{-1} \boxtimes \chi_1^{-1} \\ abab & (t_1^{-1}, t_2^{-1}) & \chi_1^{-1} \boxtimes \chi_2^{-1} \end{array}$$

Let χ a character of F^{\times} such that $\chi^{2}|_{O_{F}} \neq 1$ and $\chi(\mathfrak{D}_{F}) = 1$,

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For $\mathfrak s$ and $\mathfrak t$ we have $W_{\mathfrak s}=W_{\mathfrak t}=\{1,b\}$ and

$$T_{\mathfrak{s}} = T_{\mathfrak{t}} = \psi(T) \simeq (\mathbb{C}^{\times})^2.$$

The extended quotient is

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Remark : It is the same extended quotient for $\mathfrak s$ and $\mathfrak t$

Theorem (Sally, Tadic)

Let ψ_1, ψ_2 two characters of F^{\times} . Then $\psi_1 \times \psi_2 \rtimes 1$ is reducible, if and only if

- $\psi_1^{\pm 1} = v^{\pm 1} \psi_2$
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$\mathfrak{s} = [T, \chi \boxtimes 1]$

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$(e,(z_1,z_2))$		$\psi_1 \chi \times \psi_2 \ltimes 1$	$\psi_1 \chi \times \psi_2 \ltimes 1$
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Remark : (e, (z, -1)) and (b, (z, -1)) are in the same L-packet

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point of $T_t /\!\!/ W_t$	cocharacter	constituant	representation
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$$T_t /\!\!/ W_t = (T_t /\!\!/ W_t)_{u_0} \sqcup (T_t /\!\!/ W_t)_{u_e}$$

$$(T_{\rm t}/\!\!/W_{\rm t})_{u_0} = T_{\rm s}/W_{\rm s} \sqcup T_{\rm s}^b/W_{\rm s}$$
, cocharacter $h_{u_0}(\tau) = (1,1)$
 $(T_{\rm t}/\!\!/W_{\rm t})_{u_e} = \emptyset$

Remark : (e, (z, -1)) and (b, (z, -1)) are in the same L-packet, for (e, (z, 1)) and (b, (z, 1)) is the same

The Langlands dual group of $Sp_4(F)$ is $SO_5(\mathbb{C})$, with $SO_5(\mathbb{C}) = \{g \in SL_5(\mathbb{C}), {}^tgJg = J\}$ and the maximal torus

$$\widehat{T} = \left\{ \begin{pmatrix} z_1 & & & & \\ & z_2 & & & \\ & & 1 & & \\ & & & z_2^{-1} & \\ & & & & z_1^{-1} \end{pmatrix}, z_i \in \mathbb{C}^{\times} \right\} \simeq (\mathbb{C}^{\times})^2$$

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The Lie algebra of $SO_5(\mathbb{C})$ is

$$\mathfrak{so}_{5}(\mathbb{C}) = \left\{ \begin{pmatrix} y_{1} & x_{12} & x_{13} & x_{14} & 0 \\ x_{21} & y_{2} & x_{23} & 0 & -x_{14} \\ x_{31} & x_{32} & 0 & -x_{23} & -x_{13} \\ x_{41} & 0 & -x_{32} & -y_{2} & -x_{12} \\ 0 & -x_{41} & -x_{31} & -x_{21} & -y_{1} \end{pmatrix}, x_{ij}, y_{i} \in \mathbb{C} \right\}$$

The Langlands parameters for $\chi_1 \boxtimes \chi_2$ is

The Langlands parameters for $\chi_1 \boxtimes \chi_2$ is

$$\rho(\chi_1,\chi_2): W_F \longrightarrow \widehat{T}$$

$$w \longmapsto \begin{pmatrix} \chi_1(w) & & \\ & \chi_2(w) & & \\ & & & \\ & & & \chi_2(w)^{-1} & \\ & & & & \chi_1(w)^{-1} \end{pmatrix}$$
The Langlands parameters for the irreducible subguotients of

The Langlands parameters for the irreducible subquotients of $\psi_1 \chi \times \psi_2 \rtimes 1$ are $(\rho(\psi_1 \chi, \psi_2), 0), (\rho(\psi_1 \chi, v), 0)$ and $(\rho(\psi_1 \chi, v), N)$ with

$$N = \begin{pmatrix} 0 & & & \\ & 0 & 1 & & \\ & & 0 & -1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

The Langlands parameters are also the morphisms from W_F' to $SO_5(\mathbb{C})$. For $(\rho(\psi_1\chi, v), N)$ the corresponding morphism is

$$\phi: W_F' \longrightarrow \widehat{T} \\
(w,s) \longmapsto \begin{pmatrix} (\psi_1 \chi)(w) & & \\ & S_3(s) & \\ & & (\psi_1 \chi)(w)^{-1} \end{pmatrix}$$

where S_3 is the irreducible representation of dimension 3 of $SL_2(\mathbb{C})$.

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where S_3 is the irreducible representation of dimension 3 of $SL_2(\mathbb{C})$.

We have
$$\varphi\left(1,\begin{pmatrix} \tau^{-1} & \\ & \tau \end{pmatrix}\right) = \begin{pmatrix} 1 & & & \\ & \tau^{-2} & & \\ & & 1 & \\ & & & \tau^2 & \\ & & & 1 \end{pmatrix}$$

For the irreducible subquotient of $\psi_1 \chi \times \psi_2 \zeta \rtimes 1$ there are $(\rho(\psi_1 \chi, \psi_2 \zeta), 0)$.

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For the irreducible subquotient of $\psi_1 \chi \times \psi_2 \zeta \rtimes 1$ there are $(\rho(\psi_1 \chi, \psi_2 \zeta), 0)$.

We find in this way the cocharacters and the decomposition.

$$\mathfrak{s} = [T, \chi \boxtimes 1]$$

point of $T_{\mathfrak{S}}/\!\!/W_{\mathfrak{S}}$	cocharacter	Langands parameter	constituant	representation
$(e,(z_1,z_2))$		$(\psi_1 \chi, \psi_2)_{\widehat{T}}$	$\psi_1 \chi \times \psi_2 \ltimes 1$	$\psi_1 \chi \times \psi_2 \ltimes 1$
$(e,(z,q^{-1}))$		$(\psi\chi, v)_{\widehat{T}}$	$\psi \chi \rtimes 1_{SL_2}$	$\psi \chi \times v \ltimes 1$
(b, (z, 1))		$\begin{pmatrix} (\psi_1 \chi)(w) & & & \\ & S_3(s) & & \\ & & (\psi_1 \chi)(w)^{-1} \end{pmatrix}$	$\psi \chi \rtimes St_{SL_2}$	
(e,(z,-1))		$(\psi \chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_1^{\varepsilon}$	$\psi \chi \times \varepsilon \ltimes 1$
(b, (z, -1))		$(\psi\chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_2^{\varepsilon}$	

$$\mathfrak{s} = [T, \chi \boxtimes 1]$$

point of $T_5 /\!\!/ W_5$	cocharacter	Langands parameter	constituant	representation
$(e,(z_1,z_2))$	(1, 1)	$(\psi_1 \chi, \psi_2)_{\widehat{T}}$	$\psi_1 \chi \times \psi_2 \ltimes 1$	$\psi_1 \chi \times \psi_2 \ltimes 1$
$(e,(z,q^{-1}))$	(1, 1)	$(\psi\chi, v)_{\widehat{T}}$	$\psi \chi \rtimes 1_{SL_2}$	$\psi \chi \times v \ltimes 1$
(b, (z, 1))		$\begin{pmatrix} (\psi_1 \chi)(w) & & \\ & S_3(s) & \\ & & (\psi_1 \chi)(w)^{-1} \end{pmatrix}$	$\psi \chi \rtimes St_{SL_2}$	
(e,(z,-1))	(1, 1)	$(\psi \chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_1^{\varepsilon}$	$\psi \chi \times \varepsilon \ltimes 1$
(b, (z, -1))		$(\psi\chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_2^{\varepsilon}$	

$$\mathfrak{s} = [T, \chi \boxtimes 1]$$

	point of $T_5 /\!\!/ W_5$	cocharacter	Langands parameter	constituant	representation
_	$(e,(z_1,z_2))$	(1, 1)	$(\psi_1 \chi, \psi_2)_{\widehat{T}}$	$\psi_1 \chi \times \psi_2 \ltimes 1$	$\psi_1 \chi \times \psi_2 \ltimes 1$
	$(e,(z,q^{-1}))$	(1, 1)	$(\psi\chi, v)_{\widehat{T}}$	$\psi \chi \rtimes 1_{SL_2}$	$\psi \chi \times v \ltimes 1$
-	(b, (z, 1))	$(1, \tau^{-2})$	$\begin{pmatrix} (\psi_1 \chi)(w) & & \\ & S_3(s) & \\ & & (\psi_1 \chi)(w)^{-1} \end{pmatrix}$	$\psi \chi \rtimes St_{SL_2}$	
	(e,(z,-1))	(1, 1)	$(\psi \chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_1^{\varepsilon}$	$\psi \chi \times \varepsilon \ltimes 1$
-	(b, (z, -1))		$(\psi\chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_2^{\varepsilon}$	

$$\mathfrak{s} = [T, \chi \boxtimes 1]$$

point of $T_5 /\!\!/ W_5$	cocharacter	Langands parameter	constituant	representation
$(e,(z_1,z_2))$	(1, 1)	$(\psi_1 \chi, \psi_2)_{\widehat{T}}$	$\psi_1 \chi \times \psi_2 \ltimes 1$	$\psi_1 \chi \times \psi_2 \ltimes 1$
$(e,(z,q^{-1}))$	(1, 1)	$(\psi\chi, v)_{\widehat{T}}$	$\psi \chi \rtimes 1_{SL_2}$	$\psi \chi \times v \ltimes 1$
(b, (z, 1))	$(1, \tau^{-2})$	$\begin{pmatrix} (\psi_1 \chi)(w) & & & \\ & S_3(s) & & \\ & & (\psi_1 \chi)(w)^{-1} \end{pmatrix}$	$\psi \chi \rtimes St_{SL_2}$	
(e,(z,-1))	(1, 1)	$(\psi\chi, \varepsilon)_{\widehat{T}}$	$\psi \chi \rtimes T_1^{\varepsilon}$	$\psi \chi \times \varepsilon \ltimes 1$
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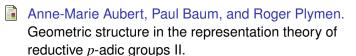
$$\mathfrak{t}=[T,\chi\boxtimes\zeta]$$

point of $T_t /\!\!/ W_t$	cocharacter	Langands parameter	constituant	representation
$(e,(z_1,z_2))$	(1, 1)	$(\psi_1\chi,\psi_2)_{\widehat{T}}$	$\psi_1 \chi \times \psi_2 \ltimes 1$	$\psi_1 \chi \times \psi_2 \ltimes 1$
(e,(z,1))	(1, 1)	$(\psi\chi,\zeta)_{\widehat{T}}$	$\psi \chi \rtimes T_1^{\zeta}$	$\psi\chi \times \zeta \ltimes 1$
(b, (z, 1))	(1, 1)	$(\psi\chi,\zeta)_{\widehat{T}}$	$\psi \chi \rtimes T_2^{\zeta}$	
(e,(z,-1))	(1, 1)	$(\psi\chi, \varepsilon\zeta)_{\widehat{T}}$	$\psi \chi \rtimes T_{1}^{\varepsilon \zeta}$	$\psi\chi \times \varepsilon\zeta \ltimes 1$
(b, (z, -1))	(1, 1)	$(\psi\chi,\varepsilon\zeta)_{\widehat{T}}$	$\psi \chi \rtimes T_2^{\varepsilon \zeta}$	

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Thank you for your attention.



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