Representations of p-adic groups Problem Sheet Number 1.

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A remark on different kinds of problems.

The problems without marking are just simple exercises. Be sure you can do them. For problems we use marking [**P**] and $(*)$ for the more difficult ones and you are advised to write the solution. The sign (∇) marks more challenging and more interesting problems which are related to some interesting subjects. It advised to think about these problems.

For an l-space X we consider the algebra $C^\infty(X)$ of locally constant functions on X. We always consider C with a discrete topology, so $C^\infty(X)$ is just an algebra of continuous functions on X. We denote by $S(X) := C^\infty(X)_c$ the subalgebra of functions with compact support. This last algebra will play an important role in the theory.

Similarly, given a vector space V over C we consider it in discrete topology and denote by $C^\infty(X, V)$ the space of continuous functions on X with values in V and by $S(X, V)$ the subspace of functions with compact support.

1. (i) Show that $S(X, V) = S(X) \otimes V$

(ii) Show that $S(X \times Y) = S(X) \otimes S(Y)$

(iii) Explain why the analogous statements are not true for C^∞ spaces.

For an l-space X we define the space of distributions $Dist(X)$ to be the complete dual space $S(X)^*$. (Standard notation $E(\phi) := \int_X \phi E$). The space $Dist(X)$ is naturally a module over the algebra $S(X)$ (in fact even over the algebra $C^\infty(X)$).

2. (i) Let K be a compact l-group. Show that there exists unique K-invariant distribution e_K on K such $\int_K e_K = 1$. Show that it is also right K-invariant. Moreover, if K acts on an ℓ space X transitively then there exists a unique K-invariant distribution on X.

(ii) Let G be an l-group. Show that there exists unique up to scalar G-invariant distribution μ_G on G. Moreover, it could be chosen to be positive (i.e it maps positive functions to positive numbers).

Usually we will fix one such positive distribution μ_G. It is called **Haar measure on** G. It is defined uniquely up to positive scalar.

(iii) We denote by $\mathcal{H}(G)$ the space of locally constant distributions with compact support. Show that a choice of Haar measure μ_G defines an isomorphism of vector spaces $S(G) \to \mathcal{H}(G)$ by $\phi \mapsto \phi\mu_G$.

[**P**] **3.** Let $Dist(X)_c$ denote the space of distributions on X with compact support.

(i) Let G be an l-group. Given a distribution $E \in Dist(G)_c$ with compact support describe its left convolution action on the space $S(G)$, $(E, \phi) \mapsto E * \phi$. Consider the adjoint right convolution on the space of distributions $(E', E) \mapsto E' * E$.

Show that this defines a convolution operation on the space $Dist(G)_c$ that defines a structure of associative algebra on the space $Dist(G)_c$. Describe the unit element of this algebra. Finally, describe the map $\delta : G \to Disc(G)_c$ which is a multiplicative embedding.

(ii) Show that $\mathcal{H}(G)$ is a subalgebra (in fact a two sided ideal) of the algebra $Dist(G)_c$.

(iii) Show that for any smooth representation (π, G, V) the action of G on V naturally extends to the action of the algebra $Dist(G)_c$.

Idempotented algebras and non-degenerate modules

Definition. (i) Let A be an associative **C**-algebra of countable dimension. We say that A is an **Idempotented algebra** if it satisfies the following condition

(idem) There exists a sequesnce of idempotent element $e_i \in A$ such that for any element $a \in A$ we have $e_i a = a = a e_i$ for large enough i.

This is a slight generalization of the notion of unital algebra (i.e. algebra that has a unit element).

(ii) If A is an idempotented algebra then an A-model M is called non-degenerate if $AM = M$. In other words this means that for any element $m \in M$ we have $e_i m = m$ for large i.

The category of non-degenerate A-modules we denote by $\mathcal{M}(A)$.

4. Let A be an idempotented algebra. Fix an idempotent $e \in A$ and consider the unital algebra $B = eAe$ (we usually denote this algebra by A_e).

(i) Show that the module $P = Ae$ is a projective object of the category $\mathcal{M}(A)$.
(ii) Show that its endomorphism algebra $End(P)$ is naturally isomorphic to the algebra B^0 opposite of B.

5. (Continuation of previous problem) Define functors $R : \mathcal{M}(A) \to \mathcal{M}(B)$ and $I : \mathcal{M}(B) \to \mathcal{M}(A)$ by $R(M) = eM = Hom(P, M)$ and $I(N) = P \otimes_B N$

(i) Show that the functor I is left adjoint to R.

(ii) Show that the functor I is right exact and functor R is exact.

(iii) Show that the canonical morphism $N \to RI(N)$ is an isomorphism.

6. (Continuation of previous problem)

(i) Assume L is a simple (=irreducible) A-module. Show that the module $R(L)$ is either 0 or irreducible.

(ii) Show that this defines a bijection of the set $Irr(B)$ with the subset $Irr(A)_e$ consisting of A-modules L such that $eL \neq 0$.

(iii) What further assumption on e is required in order to have an equivalence between categories $\mathcal{M}(A)$ and $\mathcal{M}(B)$?

7. Show that the algebra $\mathcal{H}(G)$ is idempotented (consider idempotents e_K corresponding to open compact subgroups of G.)

Show that the natural functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$ is an equivalence of categories.

8. (**Kaplansky's trick.**)

(i) Let $T : V \to V$ be an endomorphism of a complex vector space $V \neq \{0\}$. By definition the **spectrum** of T is the set of numbers $\lambda \in \mathbf{C}$ such that the operator $T - \lambda$ is not invertible. Prove the following general result from linear algebra.

(*) Assume that $\dim_{\mathbf{C}} V$ is countable. Let $T : V \to V$ be an endomorphism. Then T has a non-empty spectrum.

Hint: Consider V as a $\mathbf{C}[T]$-module. Show that if $Spec(T) = \emptyset$ then V admits a structure of a $\mathbf{C}(T)$-module.

(ii) Deduce the following corollary.

Let $T : V \to V$ as before. Then Spec(T) $= \{0\}$ iff the operator T is locally nilpotent, i.e. for every vector $v \in V$ we have that $T^n(v) = 0$ for large n.

Hint: Consider the localization of V at T and apply (i).

[P] 9. Prove the following results

Let A be as above (an idempotented algebra over $\mathbf{C}$).

(i) Schur's lemma: Let L be an irreducible A-module. Then $End_A(L) = \mathbf{C}$.

(ii) The Jacobson radical of A is defined by

$$Jac(A) = \cap_{(L,\rho) \in Irr(A) - \{0\}} Ker(\rho : A \to End(L))$$

Show that the Jacobson radical of A is contained in the set of nilpotent elements. More specifically, given $a \in A$ that is not nilpotent (i.e. all powers a^n are not zero.) Show that there exists an irreducible A-module L such that $aL \neq 0$.

[P] 10. Show that $Jac(\mathcal{H}(G)) = 0$ (the Jacobson radical is zero).