Loop spaces of classifying spaces of p-local groups as algebraic objects.

<u>Motivation</u>: Let G be a finite, or more generally, a compact Lie group. The study of p-local group theory arose from interest in the homotopy theory of spaces of the form $BG^\wedge_p$.

<u>Ex</u> $G = \Sigma_3 \cong C_3 \rtimes C_2$. What can be said about $BG^\wedge_p$.

$$BG^\wedge_p \simeq \begin{cases} * & \text{if } p > 3 \\ BC_2 & \text{if } p = 2 \end{cases}$$

For $p = 3$ $\quad H^*(BG, \mathbb{F}_3) \cong P[u_4] \otimes E[x_3]$

$\mathcal{P}^1$

So $BG^\wedge_3$ is 1-conn & there is a map $\alpha : S^3 \longrightarrow BG^\wedge_3$ inducing iso on $H^3(-, \mathbb{F}_3)$.

The following is a fibre sequence

$$S^3 \to \Omega BG_3^\wedge \to S^3 \xrightarrow{[3]} S^3 \xrightarrow{\alpha} BG_3^\wedge$$

We know that $\mathrm{hofib}(\alpha) \simeq S^3$ by an elementary spectral seq. calculation and that the fibre inclusion is $[3]$ using the Bockstein relation.

$\therefore \; \Omega BG_3^\wedge \simeq S^3\{3\}$.

The space $\Omega BG_p^\wedge$ has many interesting homotopy theoretic properties.

<u>Thm</u>: Let G be a finite group with $|Syl_p(G)| = p^m$. Then

$$p^m \cdot \widetilde{H}_*(\Omega BG_p^\wedge, \mathbb{Z}_p^\wedge) = 0$$

<u>Rk</u> $H^*(\Omega BG_p^\wedge, \mathbb{F}_p)$ is q.f. (quasi finite) and in all known examples $H_*(\Omega BG_p^\wedge, \mathbb{F}_p)$ is f.g. (finitely generated).

Note

$$G \longmapsto BG \longrightarrow BG^\wedge_p \xrightarrow{\Omega} \Omega BG^\wedge_p \xrightarrow{H_*} H_*(\Omega BG^\wedge_p; \mathbb{F}_p)$$

is a functor from Groups to $\mathbb{F}_p$-cocommutative Hopf algebras. It is thus natural to ask whether this functor can be defined algebraically.

p-compact groups

A p-compact group is a pair (X, BX) s.t. X is a loop space with a p-complete classifying space BX & $H^*(X, \mathbb{F}_p)$ is finite.

Ex All compact Lie groups G with $\pi_0 G$ a p-group, $BG^\wedge_p$ is the classifying space of a p-compact group.

There are also exotic examples.

p-local compact groups.

Algebraic (rather than homotopy theoretic) objects designed to encode the p-local homotopy theory of compact Lie groups, p-compact groups and other spaces with similar properties.

Observation p-local compact groups is a much larger class of objects than p-compact groups. Given the importance of the latter it is useful to have a good distinguishing criterion. If we could find a criterion on a p-local cpct group for when the loop space of its classifying space has finite homology (5)

we would be done.

Finite groups (Benson's work)

Let G be finite & k a field of char p.

O^pG max p-perfect normal sg. of G.

M a f.g. kG-module.

Define $[O^pG, M]$ to be the kernel of the

natural map $M \longrightarrow M \otimes_{k[O^pG]} k$.

Benson defines inductively. $P_0 = N_0 = P(k)$ -

the projective cover of the trivial $k[G]$-module k.

For $i \geq 1$ $M_{i-1} = [O^pG, N_{i-1}]$, $P_i = P(M_{i-1})$

$N_i = \mathrm{Ker}(P_i \longrightarrow M_{i-1}) \overset{def}{=} \Omega M_{i-1}$.

We get a complex

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[dr, two heads] & & P_1 \arrow[r] \arrow[d, two heads] & P_0 \arrow[r] \arrow[dl, equal] & 0 \\
\cdots & & M_1 \hookrightarrow N_1 \arrow[ur] & M_0 \hookrightarrow N_0 & &
\end{tikzcd}$$

The complex P_* is called the left squeezed resolution for G.

$$H_*(P_*) \stackrel{\text{def}}{=} H_*^{\Omega}(G,k).$$

<u>Thm [Benson]</u> $H_i^{\Omega}(G,k) \cong H_i(\Omega BG^{\wedge}_p, k) \ \forall i \geq 0.$

Let f be the idempotent in kG s.t. $f kG$ is the projective cover of the trivial module. Let $e = 1-f$.

<u>Thm [Benson]</u> For $n \geq 2$

$$H_n^{\Omega}(G,k) \cong \mathrm{Tor}_{n-1}^{e \cdot kG \cdot e}(kG \cdot e, e\,kG)$$

& there is a SES

$$0 \to H_1^{\Omega}(G,k) \to kGe \otimes_{ekGe} ekG \to kG \to H_0^{\Omega}(G,k) \to 0$$

<u>Observations</u>: Let G be a finite group. $BG \to BG^\wedge_p$ the localisation map. A^pG its homotopy fibre, the p-acyclic space associated to BG, with $\pi_1 A^pG =$ the p-universal central extension of O^pG. Consider

$$\begin{tikzcd} L^pG \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] & \\ A^pG \arrow[r] & BG \arrow[r] & BG^\wedge_p \end{tikzcd}$$

Then $L^pG \simeq \Omega BG^\wedge_p$, it has a free action of G with orbit space A^pG (the action of $O^p(G)$ on $H_*(L^pG, \mathbb{F}_p)$ is trivial. This motivates

<u>Def</u>: A left κ-squeezed resolution for G is a ~~exact~~ complex $\to P_2 \to P_1 \to P_0 \to 0$ of projective $\kappa[G]$ modules s.t.

i) $P_* \otimes_{\kappa G} \kappa$ is acyclic

ii) $[O^pG, H_*(P_*)] = 0$

Cor $C_*(L^iG, k)$ is a k-squeezed resolution for G.

p-local cpct groups Triples $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$.

$$B\mathcal{G} \overset{\text{def}}{=} |\mathcal{L}|_p^\wedge .$$

We want an algebraic model for

$$H_*(\Omega B\mathcal{G}, \mathbb{F}_p).$$

Def : G a group, $\mathcal{C}$ a small category. We say that $\mathcal{C}$ is a category over G if $\exists\, \Theta : \mathcal{C} \to BG$ $\forall c \in \mathrm{Obj}\,\mathcal{C}$ $\forall g \in G$ $\exists\, \varphi \in \mathrm{Mor}_{\mathcal{C}}(c, -)$ s.t. $\Theta(\varphi) = g$.

Def : $\mathcal{C}$ a small cat. a $k\mathcal{C}$-module is a functor $M : \mathcal{C} \to \mathrm{Vect}_k$.
$M_{\mathcal{C}} \overset{\text{def}}{=} \operatorname{colim}_{\mathcal{C}} M$. The module M is locally constant if it takes all morphisms in $\mathcal{C}$ to isos in Vect_k.

A locally constant module is locally trivial if $\forall\, p,q \in \mathcal{C}$ & all $\varphi, \psi \in \mathrm{Mor}_{\mathcal{C}}(p,q)$, $\varphi_* = \psi_*$ in $\mathrm{Iso}(M(p), M(q))$.

<u>Def:</u> Let $\theta : \mathcal{C} \to \mathcal{B}G$ be a cat. over G & let $\mathcal{C}_0 = \mathrm{Ker}(\theta)$ (= same objects with $\theta^{-1}(1)$ as morphisms). A left κ-squeezed resolution for $(\mathcal{C}, \theta)$ is a complex of projective $\kappa\mathcal{C}$-modules

$$\begin{tikzcd} \cdots P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\varepsilon"] & \underline{\varepsilon}(\kappa G) \\ & \mathcal{C} \arrow[r, "\theta"] & \mathcal{B}G \arrow[r, "\kappa(-)"] & \mathrm{Vect}_\kappa \arrow[u, squiggly] \end{tikzcd}$$

s.t.

i) $(P_*)|_{\mathcal{C}}$ & $(P_*)|_{\mathcal{C}_0}$ are acyclic & ε induces iso.s $H_0((P_*)|_{\mathcal{C}}) \cong \kappa$, $H_0((P_*)|_{\mathcal{C}_0}) \cong \kappa G$

ii) $\forall\, n$, $H_n(P_*)$ is locally constant on $\mathcal{C}$ & locally trivial on $\mathcal{C}_0$.

<u>Prop:</u> Squeezed resolutions are unique up to chain equivalence.

$E\mathcal{C} : \mathcal{C} \to Sp.$

$E\mathcal{C}(d) = |1_{\mathcal{C}} \downarrow d|$ contractible space for each object. Generalizing EG.

$$f : X \to |\mathcal{C}| \quad \text{any map.}$$

$$X \xrightarrow{f} |\mathcal{C}| \longleftarrow E\mathcal{C}(d)$$

with pullback $E_f(d)$.

* If f is a fibration then $\forall n \geq 0$ $C_n(E\mathcal{C})$ & $C_n(E_f)$ are projective $k\mathcal{C}$-modules.

* $\operatorname{hocolim}_{\mathcal{C}} E_f \simeq X$.

<u>Cor:</u> $H_*(C_*(E_f)_{\mathcal{C}}) \cong H_*(X)$.

<u>Pf:</u> SS argument using projectivity.

setup: $\theta : \mathcal{C} \to \mathcal{B}G$ cat. / G.

$\mathcal{C}_0 = \mathrm{ver}(\theta)$. assume $\pi_1(|\mathcal{C}|^\wedge_p) \cong G$

let $A\mathcal{C} = \mathrm{hofib}(|\mathcal{C}| \to |\mathcal{C}|^\wedge_p)$, & let $\iota : A\mathcal{C} \to |\mathcal{C}|$ denote the fibre inclusion

$$\begin{tikzcd}
E_\iota \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E\mathcal{C} \arrow[d] & \\
A\mathcal{C} \arrow[r, "\iota", "\text{principal fib}"'] & {|\mathcal{C}|} \arrow[r] & {|\mathcal{C}|^\wedge_p}
\end{tikzcd}$$

prop: $C_*(E_\iota)$ is a squeezed resolution for $(\mathcal{C}, \theta)$.

Cor: If small cat $\mathcal{C}$ (conn.) s.t. $\overbrace{\pi_1(|\mathcal{C}|^\wedge_p)}^{G \,\|}$ is a finite p-grp, let $\theta : \mathcal{C} \to \mathcal{B}G$ denote the obvious projection, & assume that $(\mathcal{C}, \theta)$ is a cat / G. Then $H_*(\Omega|\mathcal{C}|^\wedge_p, k) \cong H_*(P_*)(c)$ for any squeezed res. P_* & any object $c \in \mathcal{C}$.