Local similarity groups and ℓ^2-homology

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joint work with Roman Sauer

Motivation

Zero-in-the-spectrum conjecture (Gromov 86')

Let M be a closed aspherical Riemannian manifold. Then there exists always a $p \in \mathbb{N}$ such that zero is in the spectrum of the Laplacian

$$\Delta_p : \mathrm{dom}(\Delta_p) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$$

acting on square integrable p-forms on the universal covering.

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- strong Novikov conjecture $\Longrightarrow$ zero-in-the-spectrum conjecture
- false if aspherical is dropped (Farber, Weinberger 01')
- true for 2,3-manifolds, locally symmetric spaces, Kähler hyperbolic, $\sec(M) \leq 0$, $\mathrm{asdim}(\pi_1 M) < \infty$

Algebraic version

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Answer (Sauer, T. 13')

No!

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- γ local similarity:
  $\forall x \in X : \exists A, B$ balls : $x \in A, \gamma(A) = B, \gamma : A \to B$ similarity

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Examples (Nekrashevych-Röver groups, V)

...

In 2012, Dan Farley and Bruce Hughes showed that, under suitable conditions on Sim, the groups $\Gamma = \Gamma(\mathrm{Sim})$ are of type F_∞.

Vanishing of ℓ^2-homology

Definition

Sim is called dually contracting if there is a dually contracting ball in X. A ball DC is called dually contracting if there are disjoint subballs B_1, B_2 of DC and similarities $DC \to B_i$ in Sim.

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2. If $H < G$ then $H_k(H, M(H)) = 0 \implies H_k(H, M(G)) = 0$.
3. If $H_p(G_i, M(G_i)) = 0$ for $p \leq n_i$ and $i = 1, 2$ then $H_p(G_1 \times G_2, M(G_1 \times G_2)) = 0$ for $p \leq n_1 + n_2 + 1$.

Theorem (Sauer, T. 13')

If X is a compact ultrametric space with dually contracting similarity structure Sim and M nice, then for $\Gamma = \Gamma(\mathrm{Sim})$ we have

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- $H < G$ then $LH \subset LG$ flat ring extension. ($\Rightarrow$ 2)
- $LG_1 \otimes_{\mathbb{Z}} LG_2 \subset L(G_1 \times G_2)$ ring extension. ($\Rightarrow$ 3)

Sketch of proof

Fix a dually contracting ball DC. Using ping-pong lemma one can show that $\Gamma(\mathrm{Sim}|_{DC})$ contains a free non-abelian subgroup and is therefore non-amenable.

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Define simplicial Γ-complex Z via a poset $(\mathbb{P}, \leq)$:

- objects: $\mathcal{P} = \{P_1, ..., P_k\}$ partition of X into non-empty open closed subspaces ($=$ finite union of balls)
- $\leq$: $\mathcal{P} \leq \mathcal{Q}$ if $\mathcal{Q}$ refines $\mathcal{P}$, i.e. for all $Q \in \mathcal{Q}$ there is a $P \in \mathcal{P}$ with $Q \subset P$

action: $g\{P_1, ..., P_k\} := \{g(P_1), ..., g(P_k)\}$
one can show: $\mathbb{P}$ is directed $\Longrightarrow$ Z is contractible

Now let $n \in \mathbb{N}$. Define a Γ-subcomplex $Z_n \subset Z$ via a subpost $(\mathbb{P}_n, \leq)$:

$$\mathbb{P}_n := \{\mathcal{P} \in \mathbb{P} \mid \text{at least } n \text{ elements } P \in \mathcal{P} \text{ satisfy } \mathrm{Sim}(DC, P) \neq \emptyset\}$$

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Now there is a spectral sequence E^k_{pq} with

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, M(\Gamma)) \Rightarrow H_{p+q}(\Gamma, M(\Gamma))$$

Let $\sigma = \mathcal{P}_1 < ... < \mathcal{P}_p$ a cell, then observe the normal subgroup of the stabilizer group of σ

$$\Lambda_\sigma := \{g \in \Gamma \mid g(P) = P \ \forall P \in \mathcal{P}_p\} \lhd \Gamma_\sigma$$

But $\Lambda_\sigma \cong \prod_{P \in \mathcal{P}_p} \Gamma(\mathrm{Sim}|_P)$. By definition of $\mathbb{P}_n$, at least n of the $\Gamma(\mathrm{Sim}|_P)$ are isomorphic to $\Gamma(\mathrm{Sim}|_{DC})$ and consequently

$$H_0\big(\Gamma(\mathrm{Sim}|_P), M(\Gamma(\mathrm{Sim}|_P))\big) = 0$$

by 1. From 3 it follows

$$H_q(\Lambda_\sigma, M(\Lambda_\sigma)) = 0 \;\; \forall q \in \{0, ..., n-1\}$$

By 2 we have then

$$H_q(\Lambda_\sigma, M(\Gamma)) = 0 \;\; \forall q \in \{0, ..., n-1\}$$

The Hochschild-Lyndon-Serre spectral sequence yields $H_q(\Gamma_\sigma, M(\Gamma)) = 0$ in that range and the spectral sequence from above yields $H_i(\Gamma, M(\Gamma)) = 0$ for $i < n$. Since n was arbitrary, the result follows.

Thank you for your attention.