ℓ^2-Betti numbers for group theorists

A minicourse in 3 parts – 2nd lecture

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Copenhagen, October 2013

The von Neumann dimension $\dim_\Gamma$

Finite-dimensional vector spaces

Let $W \subset \mathbb{C}^n$ be a subspace and $\mathrm{pr}_W : \mathbb{C}^n \to \mathbb{C}^n$ be the projection onto W. Then

$$\dim_{\mathbb{C}}(W) = \mathrm{tr}_{M_n(\mathbb{C})}(\mathrm{pr}_W).$$

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The von Neumann trace

Let $L(\Gamma)$ be the algebra of Γ-equivariant bounded operators on $\ell^2(\Gamma)$ (**von Neumann algebra** of Γ). For $T \in L(\Gamma)$ define:

$$\mathrm{tr}_\Gamma(T) = \langle Te, e\rangle_{\ell^2(\Gamma)}.$$

It satisfies $\mathrm{tr}_\Gamma(ST) = \mathrm{tr}_\Gamma(TS)$ and extends to $M_n(L(\Gamma))$.

Hilbert Γ-modules

A **Hilbert Γ-module** is a Hilbert space with an isometric linear Γ-action such that there is an Γ-equivariant isometric embedding $H \hookrightarrow \ell^2(\Gamma)^n$.

$$\dim_\Gamma(H) := \mathrm{tr}_{M_n(L(\Gamma))}(\mathrm{pr}_H)$$

ℓ^2-Betti numbers

Definition

Let X be a cocompact free Γ-CW complex. Its ℓ^2-cohomology $H^i_{(2)}(X)$ is a Hilbert Γ-module via the embedding:

$$\bar{H}^i_{(2)}(X) \cong \ker(\Delta^i) \hookrightarrow \ell^2(\Gamma)^n.$$

We define the ℓ^2-**Betti numbers** as:

$$\beta_i^{(2)}(X) = \dim_\Gamma\left(\bar{H}^i_{(2)}(X)\right)$$
$$\beta_i^{(2)}(\Gamma) = \dim_\Gamma\left(\bar{H}^i_{(2)}(E\Gamma)\right)$$

Homology versus cohomology

Alternatively, we could define $\beta_i^{(2)}$ by reduced homology. The Laplace operators are the same for homology and cohomology.

Properties

ℓ^2-Betti numbers satisfy equivariant homotopy invariance, a Künneth formula, and a Euler-Poincare formula....

A question by Atiyah

These real Betti numbers appear to deserve further study. Some natural questions are :

(i) Triangulate X and compute the simplicial L^2 cohomology of $\tilde{X}$ for the lifted triangulation (using cocycles/closure of coboundaries). Are these groups Γ-isomorphic to our $\mathcal{H}^q(\tilde{X})$? †

(ii) If the answer to (i) is yes, are the $B^q_\Gamma(\tilde{X})$ homotopy invariants of X ?

(iii) A priori the numbers $B^q_\Gamma(\tilde{X})$ are real. Give examples where they are not integral and even perhaps irrational.

Conjectures

Atiyah conjecture

Let Γ be a torsionfree group. Then the ℓ^2-Betti numbers of a free cocompact Γ-CW complex are in $\mathbb{N} \cup \{0\}$.

Algebraic Atiyah conjecture

Let Γ be a torsionfree group. Let $A \in M_{m,n}(\mathbb{Z}\Gamma)$. Then

$$\dim_\Gamma\Big(\ker\big(\ell^2(\Gamma)^m \xrightarrow{-\cdot A} \ell^2(\Gamma)^n\big)\Big) \in \mathbb{N} \cup \{0\}.$$

Zero divisor conjecture

Let Γ be a torsionfree group. The group $\mathbb{Z}\Gamma$ has no zero-divisors.

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Relations

$$\text{Atiyah} \iff \text{algebraic Atiyah} \Rightarrow \text{ZD conjecture}$$

Free groups: Linnell's proof

Fredholm module

A **Fredholm Γ-module** consists of $*$-homomorphisms $\rho_\pm : L(\Gamma) \to \mathcal{B}(\mathcal{H})$ such that $\rho_+(a) - \rho_-(a)$ has finite rank for every $a \in \mathbb{C}\Gamma$.

Construct a Fredholm module with trace property

We construct a specific Fredholm F_2-module $\rho_\pm$ such that

$$\underbrace{\mathrm{tr}_{F_2}(a)}_{F_2\text{-trace}} = \underbrace{\mathrm{tr}\big(\rho_+(a) - \rho_-(a)\big)}_{\text{usual trace}} =: \tau(a) \quad \text{for all } a \in L(F_2).$$

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Let (V, E) be the **Cayley graph** of F_2. Let

$$f : V \xrightarrow{\sim} E \coprod \{pt\}$$

$$f(x) = \begin{cases} \text{first edge on geodesic from } x \text{ to } x_0 & \text{if } x \neq x_0 \\ pt & \text{otherwise} \end{cases}$$

- Standard reps $\rho_+ : L(F_2) \to \mathcal{B}(\underbrace{\ell^2(V)}_{\cong \ell^2(\Gamma)})$ and $\rho : L(F_2) \to \mathcal{B}(\ell^2(E) \oplus \mathbb{C})$
- $\rho_- := F^{-1} \circ \rho \circ F : L(F_2) \to \mathcal{B}(\ell^2(V))$ with F **induced by** f.

Free groups: Linnell's proof

Integrality lemma

Let $p, q \in \mathcal{B}(\mathcal{H})$ be projections for which $p - q$ has finite rank. Then $\mathrm{tr}(p - q) \in \mathbb{Z}$.

Proof.

p, q commute with $(p-q)^2$, thus respect the eigenspace decomposition of $(p-q)^2$:

$$\mathcal{H} = \bigoplus_{\lambda \neq 0} E_\lambda \oplus \underbrace{\ker(p-q)^2}_{=\ker(p-q)}$$

This implies: $\mathrm{tr}(p-q) = \sum_{\lambda \neq 0} \mathrm{tr}\big(p|_{E_\lambda}\big) - \sum_{\lambda \neq 0} \mathrm{tr}\big(q|_{E_\lambda}\big) \in \mathbb{Z}$ □

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Finite rank lemma

Let $a, b \in \mathcal{B}(\mathcal{H})$ such that $a - b$ has finite rank. Then $\operatorname{pr}_{\ker(a)} - \operatorname{pr}_{\ker(b)}$ has finite rank.

Proof.

- It is easy to see that $\operatorname{pr}_{\ker(a)}$ and $\operatorname{pr}_{\ker(b)}$ agree on $\ker(a-b)$.
- Since $a - b$ has finite rank, $\ker(a-b)^\perp$ is finite-dimensional. □

A few comments on the status of the conjectures

- Linnell: (Strong) Atiyah conjecture for a certain class of groups that contains free groups and elementary amenable groups with upper bounds on orders of finite subgroups.
- This builds on earlier work of Kropholler-Linnell-Moody who proved the ZD conjecture for torsionfree elementary amenable groups.
- A lot of positive results by Linnell, Schick and others...
- The Grigorchuk-Schick-Zuk counterexample to the strong Atiyah conjecture.
- Austin's counterexample to the rationality of L^2-Betti numbers.