ℓ^2-Betti numbers for group theorists

A minicourse in 3 parts – 1st lecture

Roman Sauer

Karlsruhe Institute of Technology

Copenhagen, October 2013

First impression

$$\Gamma \mapsto \quad \beta_0^{(2)}(\Gamma),\ \beta_1^{(2)}(\Gamma),\ \beta_2^{(2)}(\Gamma)\dots \qquad \in [0,\infty]$$

$$\Gamma \curvearrowright X \mapsto \quad \beta_0^{(2)}(X),\ \beta_1^{(2)}(X),\ \beta_2^{(2)}(X)\dots \qquad \in [0,\infty]$$

- For $X = E\Gamma$ we have $\beta_i^{(2)}(X) = \beta_i^{(2)}(\Gamma)$.

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$$\begin{aligned} \Gamma \mapsto \quad & \beta_0^{(2)}(\Gamma),\ \beta_1^{(2)}(\Gamma),\ \beta_2^{(2)}(\Gamma)\dots & \in [0,\infty] \\ \Gamma \curvearrowright X \mapsto \quad & \beta_0^{(2)}(X),\ \beta_1^{(2)}(X),\ \beta_2^{(2)}(X)\dots & \in [0,\infty] \end{aligned}$$

- For $X = E\Gamma$ we have $\beta_i^{(2)}(X) = \beta_i^{(2)}(\Gamma)$.
- Remember: $\chi(\Gamma) = \sum_{i\geq 0}(-1)^i \beta_i^{(2)}(\Gamma)$ and $\beta_0^{(2)}(\Gamma) = 0$ for infinite Γ.
- most important: is $\beta_i^{(2)}(\Gamma) = 0$ or not?
- For instance here: If Γ is finitely presented and residually p-finite with $\beta_1^{(2)}(\Gamma) > 0$, then Γ is large (Lackenby).

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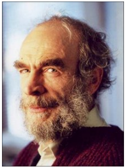
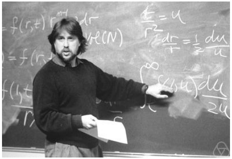
The founders





ℓ^2-cohomology

- ℓ^2-Betti numbers are **some dimension** of a certain cohomology – ℓ^2-**cohomology**: $\beta_i^{(2)}(\Gamma \curvearrowright X) = \dim_\Gamma\big(\bar{H}^i_{(2)}(X)\big)$
- We postpone the definition of $\dim_\Gamma$ for a while; important for us is:

$$\beta_i^{(2)}(\Gamma \curvearrowright X) = 0 \iff \bar{H}^i_{(2)}(X) = 0$$

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Definition

- Let X be a CW complex and $C^0(X) \xrightarrow{d} C^1(X) \xrightarrow{d} \dots$ its complex of cellular $\mathbb{C}$-valued cochains.
- Consider the sub-complex $C^0_{(2)}(X) \xrightarrow{d} C^1_{(2)}(X) \xrightarrow{d} \dots$ of **square-summable cochains** and its cohomology:

$$H^i_{(2)}(X) = \ker(d)/\operatorname{im}(d)$$
$$\bar{H}^i_{(2)}(X) = \ker(d)/\operatorname{clos}(\operatorname{im}(d))$$

ℓ^2-cohomology by harmonic cocycles

- $\Delta = dd^* + d^*d : C^i_{(2)}(X) \to C^i_{(2)}(X)$ **Laplace operator**
- The Δ's as a chain map is $\simeq$ to the zero map, i.e. there is a chain homotopy $h : C^i_{(2)}(X) \to C^{i-1}_{(2)}(X)$ with $hd + dh = \Delta - 0$.

$$H^i_{(2)}(X) = 0 \Leftarrow \Delta^i \text{ invertible}$$
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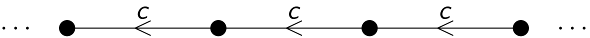
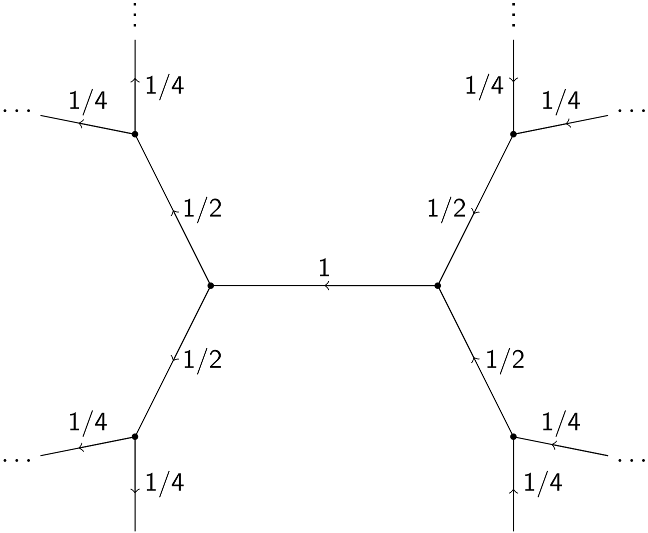
- One has $\ker(\Delta) = \ker(d) \cap \ker(d^*)$; elements in $\ker(\Delta)$ are called **harmonic**.
- $d^* : C^1_{(2)}(X) \to C^0_{(2)}(X)$ explicitly:

$$d^*f(v) = \sum_{e(+)=v} f(e) - \sum_{e(-)=v} f(e)$$

ℓ^2-cohomology by harmonic cocycles





Geometric and algebraic viewpoint

Geometric

- The definition of $\bar{H}^i_{(2)}(X)$ does not involve the Γ-action. The groups $\bar{H}^i_{(2)}(\_)$ are functorial under bi-lipschitz maps.
- Pansu: If $\Gamma \sim_{QI} \Lambda$, then $\bar{H}^i_{(2)}(E\Gamma) \cong \bar{H}^i_{(2)}(E\Lambda)$.
- Thus, the vanishing of the ith ℓ^2-Betti number is a QI-invariant.

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Algebraic

- $H^i_{(2)}(X)$ is just cohomology with twisted coefficients for a free cocompact Γ-CW complex X.
- The isomorphism $H^i_\Gamma(X, \ell^2(\Gamma)) \cong H^i_{(2)}(X)$ is induced by
$$\hom_\Gamma(C_i(X), \ell^2(\Gamma)) \xrightarrow{\cong} C^i_{(2)}(X),\ f \mapsto (\sigma \mapsto f(\sigma)(1)).$$
- If n is the number of equivariant i-cells in X, then
$$\hom_\Gamma(C_i(X), \ell^2(\Gamma)) \cong \ell^2(\Gamma)^n.$$

The von Neumann dimension $\dim_\Gamma$

Finite-dimensional vector spaces

Let $W \subset \mathbb{C}^n$ be a subspace and $\mathrm{pr}_W : \mathbb{C}^n \to \mathbb{C}^n$ be the projection onto W. Then

$$\dim_{\mathbb{C}}(W) = \mathrm{tr}_{M_n(\mathbb{C})}(\mathrm{pr}_W).$$

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The von Neumann trace

Let $L(\Gamma)$ be the algebra of Γ-equivariant bounded operators on $\ell^2(\Gamma)$ (**von Neumann algebra** of Γ). For $T \in L(\Gamma)$ define:

$$\mathrm{tr}_\Gamma(T) = \langle Te, e \rangle_{\ell^2(\Gamma)}.$$

It satisfies $\mathrm{tr}_\Gamma(ST) = \mathrm{tr}_\Gamma(TS)$ and extends to $M_n(L(\Gamma))$.

Hilbert Γ-modules

A **Hilbert Γ-module** is a Hilbert space with an isometric linear Γ-action such that there is an Γ-equivariant isometric embedding $H \hookrightarrow \ell^2(\Gamma)^n$.

$$\dim_\Gamma(H) := \mathrm{tr}_{M_n(L(\Gamma))}(\mathrm{pr}_H)$$

ℓ^2-Betti numbers

Definition

Let X be a cocompact free Γ-CW complex. Its ℓ^2-cohomology $H^i_{(2)}(X)$ is a Hilbert Γ-module via the embedding:

$$\bar{H}^i_{(2)}(X) \cong \ker(\Delta^i) \hookrightarrow \ell^2(\Gamma)^n.$$

We define the ℓ^2-**Betti numbers** as:

$$\beta_i^{(2)}(X) = \dim_\Gamma\left(\bar{H}^i_{(2)}(X)\right)$$
$$\beta_i^{(2)}(\Gamma) = \dim_\Gamma\left(\bar{H}^i_{(2)}(E\Gamma)\right)$$

Homology versus cohomology

Alternatively, we could define $\beta_i^{(2)}$ by reduced homology. The Laplace operators are the same for homology and cohomology.

Properties

ℓ^2-Betti numbers satisfy equivariant homotopy invariance, a Künneth formula, and a Euler-Poincare formula....