Construction of boundaries for non-positively curved complexes of groups

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INTRODUCTION :

COMBINATION PROBLEMS IN GEOMETRIC GROUP THEORY

COMBINATION PROBLEM : Let G be a group acting cocompactly on a simplicial complex X. What properties of G can we deduce from:

- the geometry of X,
- the dynamics of the action,
- the various inclusions of stabilisers?

Bass-Serre theory

The study of groups acting non-trivially on simplicial trees.

A few remarkable consequences:

- a geometric proof of the freeness of torsion-free discrete subgroups of $SL_2(\mathbb{Q}_p)$,
- the hyperbolicity of the fundamental group of some 3-manifolds obtained as mapping tori of hyperbolic surfaces along pseudo-Anosov diffeomorphisms (Bestvina-Feighn),
- the relative hyperbolicity of limit groups (Dahmani).

Geometric actions

The study of groups acting properly and cocompactly on simplicial complexes.

A few examples:

- cubulation of groups (Wise, Haglund, Agol),
- construction of finitely-presented torsion-free simple groups (Burger-Mozes),
- hyperbolic Coxeter groups of large virtual cohomological dimension (Januszkiewicz-Świątkowski).

SETTING OF THIS WORK : Actions which are cocompact, but not necessarily proper, on simplicial complexes endowed with a geometry which is non-positively curved in a broad sense, and more specifically on CAT(0) simplicial complexes.

Two important examples of such a situation:

- the mapping class group of a hyperbolic surface acting on its curve complex,
- the action of a group admitting a codimension one subgroup on the associated CAT(0) cube complex (Sageev).

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We will be interested in combination problems for the following properties:

- existence of a cocompact model of classifying space for proper actions,
- existence of a suitable boundary ($E\mathcal{Z}$-structure),
- hyperbolicity.

PRELIMINARIES :

COMPLEXES OF GROUPS

Definition (Gersten–Stallings, Corson, Haefliger)

Let Y be a simplicial complex. A *complex of groups* $G(\mathcal{Y}) = (G_\sigma, \psi_{\sigma,\sigma'}, \qquad)$ *over* Y consists of the following data:

- for each σ of Y, a group G_σ called a *local group*,
- for each inclusion of simplices $\sigma \subset \sigma'$, an injective morphism $\psi_{\sigma,\sigma'} : G_{\sigma'} \to G_\sigma$,

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- for each inclusion of simplices $\sigma \subset \sigma'$, an injective morphism $\psi_{\sigma,\sigma'} : G_{\sigma'} \to G_\sigma$,
- for each inclusion $\sigma \subset \sigma' \subset \sigma''$, a *twist coefficient* $g_{\sigma,\sigma',\sigma''} \in G_\sigma$,

subject to the following compatibility conditions:

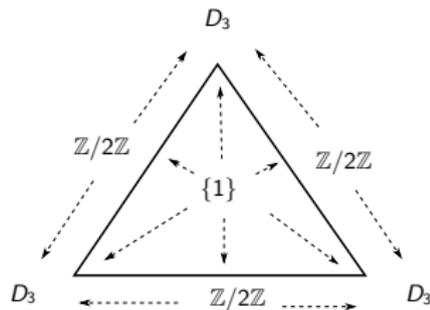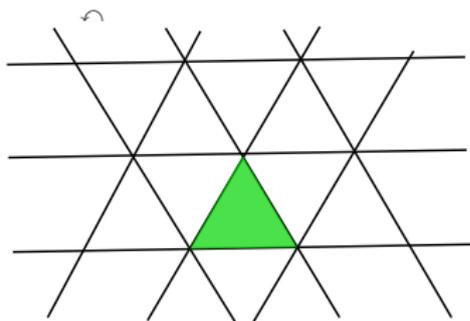- for each inclusion $\sigma \subset \sigma' \subset \sigma''$, we have

$$g_{\sigma,\sigma',\sigma''} \cdot \psi_{\sigma,\sigma''} \cdot g_{\sigma,\sigma',\sigma''}^{-1} = \psi_{\sigma,\sigma'} \circ \psi_{\sigma',\sigma''},$$

- for each inclusion $\sigma \subset \sigma' \subset \sigma'' \subset \sigma'''$, we have the following *cocycle condition*:

$$\psi_{\sigma,\sigma'}(g_{\sigma',\sigma'',\sigma'''})g_{\sigma,\sigma',\sigma'''} = g_{\sigma,\sigma',\sigma''}g_{\sigma,\sigma'',\sigma'''}.$$
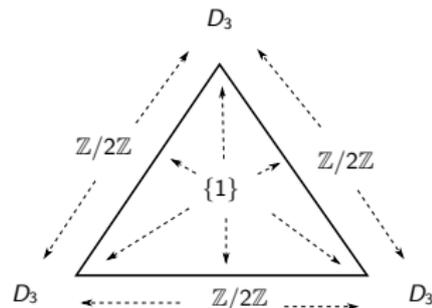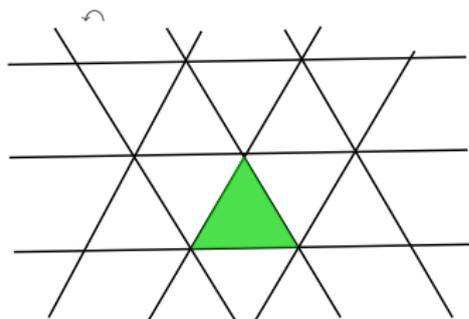
Given a group G acting without inversion on a simply-connected simplicial complex X, we can construct an oassociated complex of groups over $G \setminus X$.

$T(3,3,3) = \langle a,b,c | a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = 1 \rangle$

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A complex of groups is said to be *developable* if it comes from such an action $G \curvearrowright X$. In such a case :

- G is unique up to isomorphism (*fundamental group* of the complex of groups)
- X is unique up to equivariant simplicial isomorphism (*universal cover* of the complex of groups)

Unlike in Bass-Serre theory, not every complex of groups is developable.

Nonetheless, a complex of groups is always locally developable, non-developability being a global phenomenon (notion of a *local development*).
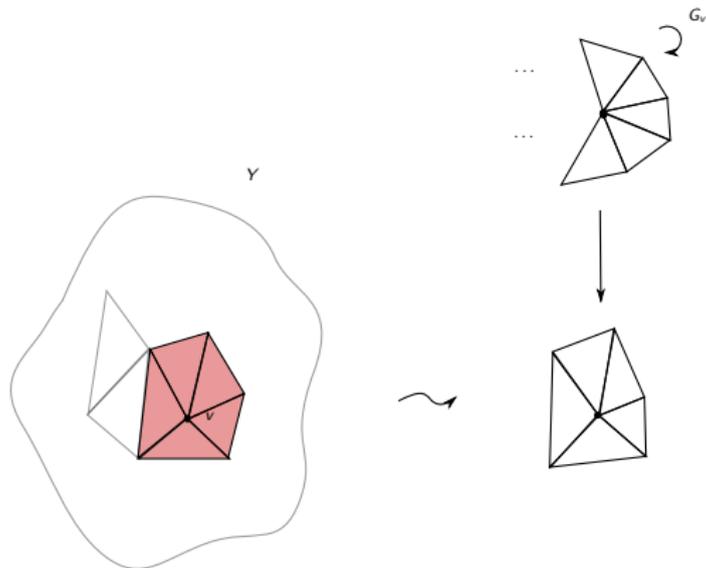
There is a geometric criterion implying developability:

Suppose that the finite complex Y is endowed with a piecewise-Euclidean structure. Such a structure yields a piecewise-Euclidean structure on each local development.

CAT(0) developability theorem (Haefliger)

If each local development is CAT(0) for the induced metric, then the complex of groups is developable and its universal cover is CAT(0).

PART I :

COMBINATION OF CLASSIFYING SPACES FOR PROPER ACTIONS

Definition

A cocompact model of classifying space for proper actions for a finitely generated group G is a CW-complex EG endowed with a properly discontinuous cocompact G-action by cellula homeomorphisms, such that for every finite subgroup H of G, the set EG^H of point fixed under H is contractible.

An important example : Let G be a hyperbolic group. Then G admits an appropriate Rips complex as a cocompact model of classifying space for proper actions.

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COMBINATION PROBLEM : Let $G(\mathcal{Y})$ be a complex of groups over a finite complex Y, such that each local group admits a cocompact model of classifying space for proper actions. Does G admit such a model?

Idea : To generalise the topological construction of Scott–Wall for graphs of groups.

A *complex of spaces* $C(\mathcal{Y})$ over a complex Y consists of the following data: for every simplex $\sigma \subset Y$, a topological space C_σ, and for every inclusion of simplices $\sigma \subset \sigma'$, a continuous map $\phi_{\sigma,\sigma'} : C_{\sigma'} \to C_\sigma$ such that the associated diagram of maps is commutative.
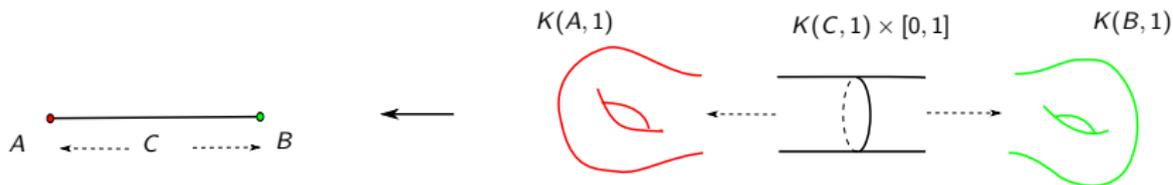
To a complex of spaces $C(\mathcal{Y})$, one can associate its *realisation*, which is the space

$$|C(\mathcal{Y})| := (\coprod_{\sigma \subset Y} \sigma \times C_\sigma \,)/ \sim,$$

où

$$(i_{\sigma,\sigma'}(x), s) \sim (x, \phi_{\sigma,\sigma'}(s)) \text{ for } x \in \sigma \subset \sigma' \text{ and } s \in C_{\sigma'},$$

and $i_{\sigma,\sigma'} : \sigma \hookrightarrow \sigma'$ is the inclusion.

Definition

Given a complex of groups over a finite complex Y, a *complex of classifying spaces* $EG(\mathcal{Y})$ *compatible with* $G(\mathcal{Y})$ consists of the following:

- for every simplex $\sigma \subset Y$, a cocompact model of classifying space for proper actions of G_σ, denoted EG_σ,
- for every inclusion $\sigma \subset \sigma'$, a $\psi_{\sigma,\sigma'}$-equivariant continuous map $\phi_{\sigma,\sigma'} : EG_{\sigma'} \to EG_\sigma$,

subject to the following compatibility condition:

- for every inclusion $\sigma \subset \sigma' \subset \sigma''$, we have

$$g_{\sigma,\sigma',\sigma''} \cdot \phi_{\sigma,\sigma''} = \phi_{\sigma,\sigma'} \circ \phi_{\sigma',\sigma''}.$$

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Remark: A complex of classifying spaces compatible with a complex of groups is not a complex of spaces when the twist coefficients are non-trivial.

Idea : From $EG(\mathcal{Y})$, one can construct a cocompact model of classifying space for proper actions of G as a complex of spaces over the universal cover X.

Proposition (Combination of classifying spaces)

Let $G(\mathcal{Y})$ be a complex of groups over a finite complex Y, with fundamental group G universal cover X. Supposose that:

- for every finite subgroup H of G, the fixed-point set X^H is contractible,
- every local group admits a cocompact model of classifying space for proper actions.
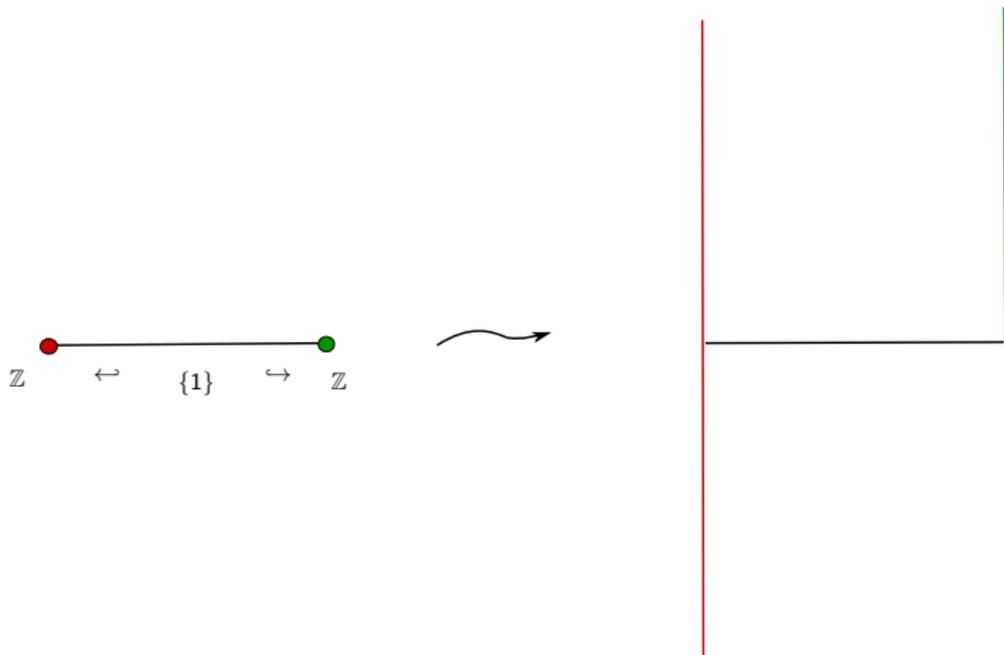
Then there exists a compatible complex of classifying spaces. In particular, G admits a cocompact model of classifying space for proper actions.

An example: the free product $G = \mathbb{Z} * \mathbb{Z}$, seen as an edge of groups.

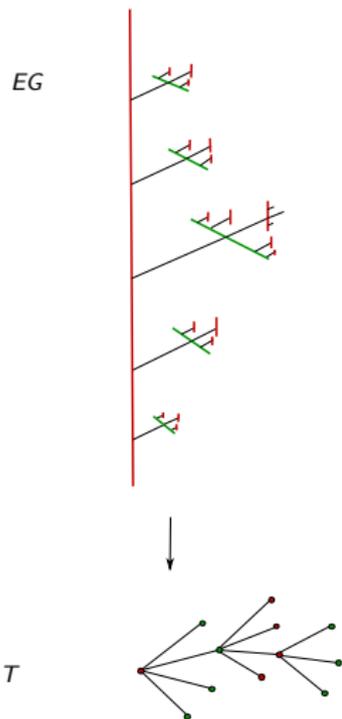$\mathbb{Z} \quad \hookleftarrow \quad \{1\} \quad \hookrightarrow \quad \mathbb{Z}$
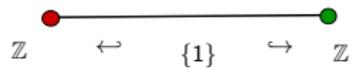
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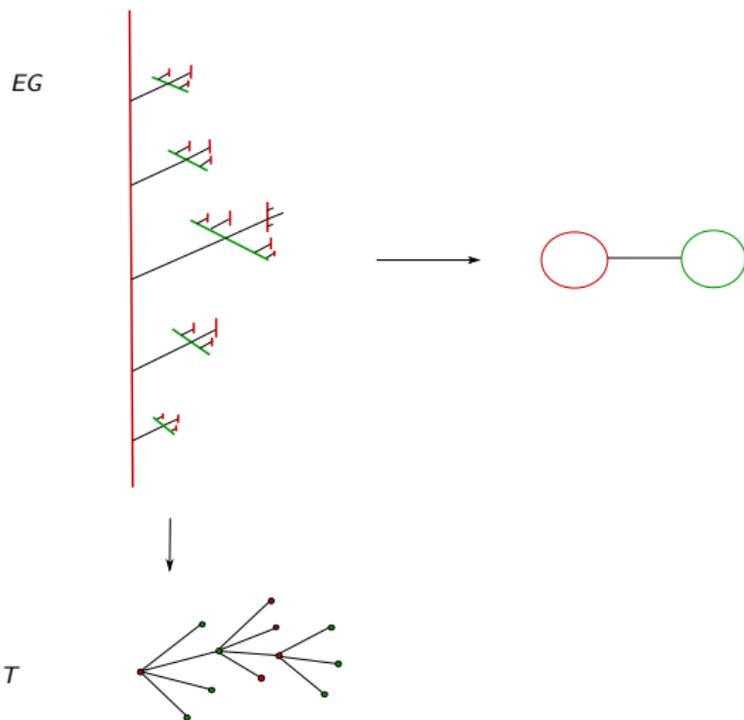
We have the following compatible complex of classifying spaces, whose realisation is:

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PART II

COMBINATION OF BOUNDARIES OF GROUPS

Definition

Let G be a discrete admitting a cocompact model EG of classifying space for proper actions. An $E\mathcal{Z}$-structure for G is a compactification $\overline{EG} = EG \cup \partial G$ of EG such that:

- $\overline{EG}$ is a compact, metrisable, contractible and locally contractible space of finite topological dimension,
- the boundary ∂G is a $\mathcal{Z}$-set in $\overline{EG}$, i.e. for every neighbourhood U in $\overline{EG}$ of a boundary point, the inclusion $U \setminus \partial G \hookrightarrow U$ is a homotopy equivalence,
- compact sets *fade at infinity*, i.e. for every $\xi \in \partial G$, for every neighbourhood U of ξ in $\overline{EG}$ and every compact subset K of EG, there exists a subneighbourhood V such that a G-translate of K meeting V is necessarily contained in U,
- the actions of G on EG continuously extends to $\overline{EG}$.

A few examples:

- Let G be a group acting geometrically on a CAT(0) simplicial complex. Then X is a cocompact model of classifying space for proper actions of G. One obtains an $E\mathcal{Z}$-structure by adding to X its visual boundary.

- Let G be a hyperbolic group. An $E\mathcal{Z}$-structure is obtained by adding to an approriate Rips complex of G the Gromov boundary of G (Bestvina-Mess, Meintrup-Schick).

- There exists similar constructions for systolic groups (Osajda-Przytycki).

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Theorem (Farrell-Lafont)

Let G be a group admitting an $E\mathcal{Z}$-structure. Then G satisfies the Novikov conjecture.

COMBINATION PROBLEM: Let $G(\mathcal{Y})$ be a developable complex of groups over a finite complex Y, such that each local group admits an $E\mathcal{Z}$-structure. Is it possible to construct an $E\mathcal{Z}$-structure for its fundamental group?

Conditions of three types:

- *geometric* condition: $G(\mathcal{Y})$ is non-positively curved, i.e. its universal cover is CAT(0),
- *dynamical* condition : the action of G on X is *acylindrical*, i.e. there exists a constant $A \geq 0$ such that every subcomplex of X of diameter at least A has a finite pointwise stabiliser,
- *algebraic* condition on the morphisms between local groups: to make things easier, we will assume that all the local groups are hyperbolic and all the local maps are quasiconvex embeddings.
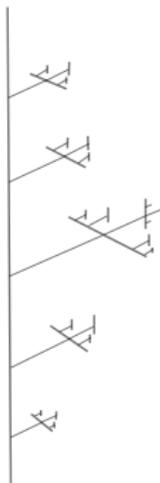
Combination theorem for boundaries of groups

Let $G(\mathcal{Y})$ be a developable complex of groups over a finite complex Y, with fundamental group G and universal cover X. Suppose that:

- the universal cover X is CAT(0),
- the action of G on X is acylindrical,
- the local groups are hyperbolic and the morphisms are quasiconvex embeddings.

Then G admits an $E\mathcal{Z}$-structure (and we have an explicit description of the boundary).

Idea of the proof: We follow a strategy of Dahmani to study graphs of relatively hyperbolic groups.
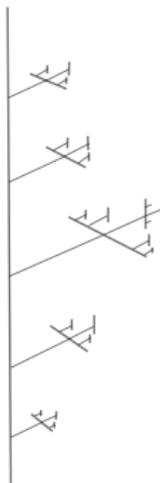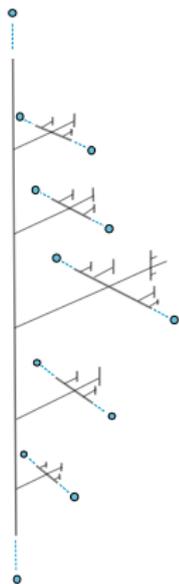
We start by defining the boundary as a set.

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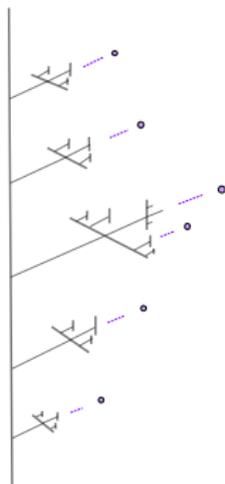

Two types of boundary points:

- points in boundaries of stabilisers of simplices of X.

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Two types of boundary points:

- points in boundaries of stabilisers of simplices of X.

- points in the visual boundary of X,

The boundary of G is defined (as a set) as the reunion

$$\partial G := \partial X \sqcup \underbrace{\Big(\coprod_{\sigma \subset X} \partial G_\sigma \Big)/ \sim}_{=: \partial_{Stab} G},$$

where for every inclusion of simplices $\sigma \subset \sigma'$, we identify $\xi \in \partial G_{\sigma'}$ and its image $\phi_{\sigma,\sigma'}(\xi) \in \partial G_\sigma$.

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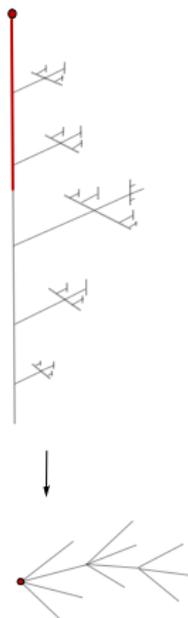
where for every inclusion of simplices $\sigma \subset \sigma'$, we identify $\xi \in \partial G_{\sigma'}$ and its image $\phi_{\sigma,\sigma'}(\xi) \in \partial G_\sigma$.

Because of these identifications, a point in $\partial_{Stab} G$ is no longer "above" a unique vertex, but above a subcomplex. More precisely:
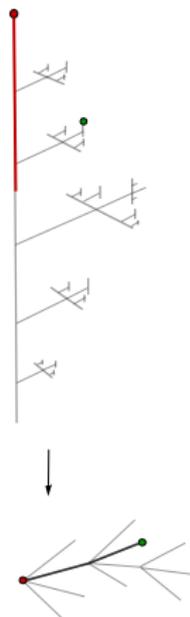
Definition

For a point $\xi \in \partial_{Stab} G$, we call *domain of ξ*, denoted $D(\xi)$, the reunion of those simplices σ of X such that ∂G_σ contains a point in the class of ξ.
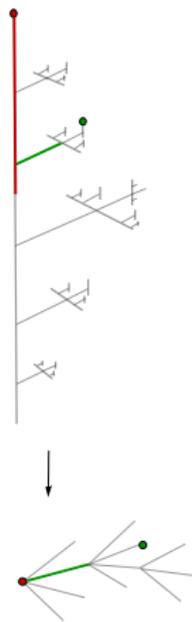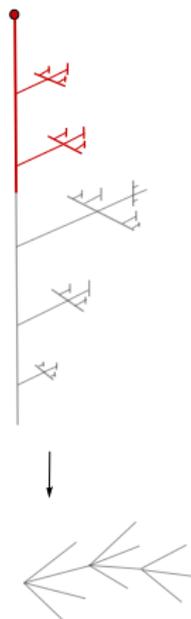
Idea to define the topology:

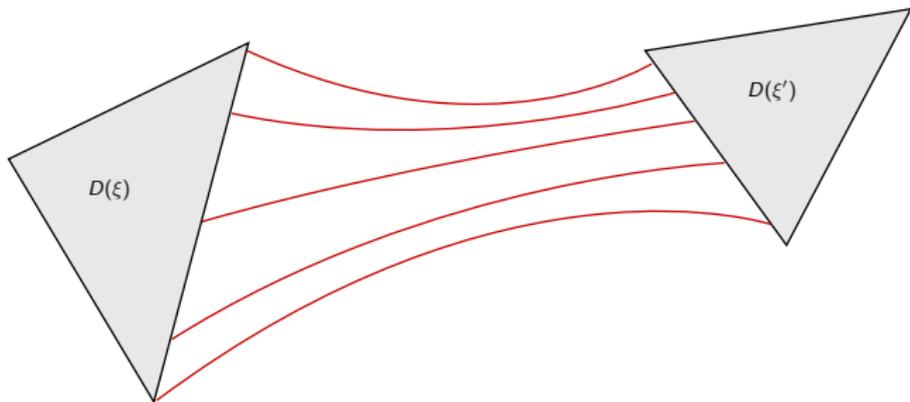**Idea to define the topology:**

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In the general case, a point ξ de $\partial_{Stab} G$ is no longer above a unique vertex, but above it domain. We are thus led to consider the set of geodesics between two domains $D(\xi)$ and $D(\xi')$.

Two remarks:

- The conditions of acylindricity, CAT(0) universal cover, and quasiconvex embeddings between local groups have the following consequence:

Proposition

Domains are convex subcomplexes with a uniformly bounded number of simplices.

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- The conditions of acylindricity, CAT(0) universal cover, and quasiconvex embeddings between local groups have the following consequence:

Proposition

Domains are convex subcomplexes with a uniformly bounded number of simplices.

- Sets of geodesics in piecewise-Euclidean complexes with finitely many isometry types of simplices have very strong finiteness properties.

PART III

COMBINATION OF HYPERBOLIC GROUPS

COMBINATION PROBLEM: Let G be a group acting cocompactly on a simply-connected hyperbolic space X, such that each simplex stabiliser is hyperbolic. Is G itself hyperbolic ?

False in general:
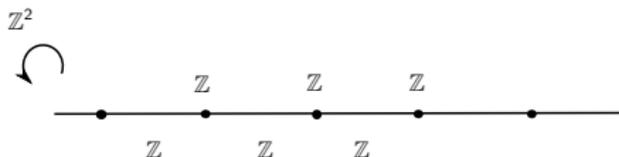
There exists such a combination theorem for graphs of groups.

Acylindrical version of the Bestvina–Feighn combination theorem

Consider a finite graph of groups with fundamental group G and Bass-Serre tree T, such that:

- the action of G on T is acylindrical,
- the local groups are hyperbolic and the morphisms are quasiconvex embeddings.

Then G is hyperbolic.

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The general theorem also proves the hyperbolicity of some fundamental groups of 3-manifolds obtained as mapping tori.

Combination theorem for hyperbolic groups

Let $G(\mathcal{Y})$ be a developable complex of groups over a finite complex Y, with fundamental group G and universal cover X. Suppose that:

- the universal cover X is CAT(0) and hyperbolic,
- the action of G on X is acylindrical,
- the local groups are hyperbolic and the morphisms are quasiconvex embeddings.

Then G is hyperbolic, and the local groups embed in G as quasiconvex subgroups.

Idea of the proof: We use the dynamical characterisation of hyperbolicity due to Bowditch.

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Theorem (Bowditch)

Let G be a finitely generated group acting by homeomorphisms on a cocompact metrisable space containing at least three points, and such that:

- for every sequence of elements (g_n) of G, there exists a subsequence $(g_{\varphi(n)})$ and points ξ_-, ξ_+ of M such that for every compact subset $K \subset M \setminus \{\xi_-\}$, the sequence of translates $g_{\varphi(n)}K$ uniformly converges to ξ_+,
- for every element ξ of M, there exists a sequence (g_n) of elements of G and two distinct points ξ_-, ξ_+ of M such that $g_n\xi \to \xi_-$ and for every element $\xi' \neq \xi$, we have $g_n\xi' \to \xi_+$.

Then G is hyperbolic, and M is G-equivariantly homeomorphic to the Gromov boundary of G.

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Then G is hyperbolic, and M is G-equivariantly homeomorphic to the Gromov boundary of G.

From the combination theorem for boundaries of groups, G already admits an $E\mathcal{Z}$-structure $(\overline{EG}, \partial G)$. We thus have a candidate for the Gromov boundary of G. It remains to study the dynamics of the action on ∂G.