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# Finiteness properties of arithmetic groups II

We are interested in $SL_n(\mathbb{F}_q[t])$

but today we will concentrate on $\underline{n=2}$ and consider $SL_2(\mathbb{Z})$ as a warm-up.

## $\underline{SL_2(\mathbb{Z})}$

$SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}^2 = \{ x+iy \in \mathbb{C} \mid y>0 \}$ via

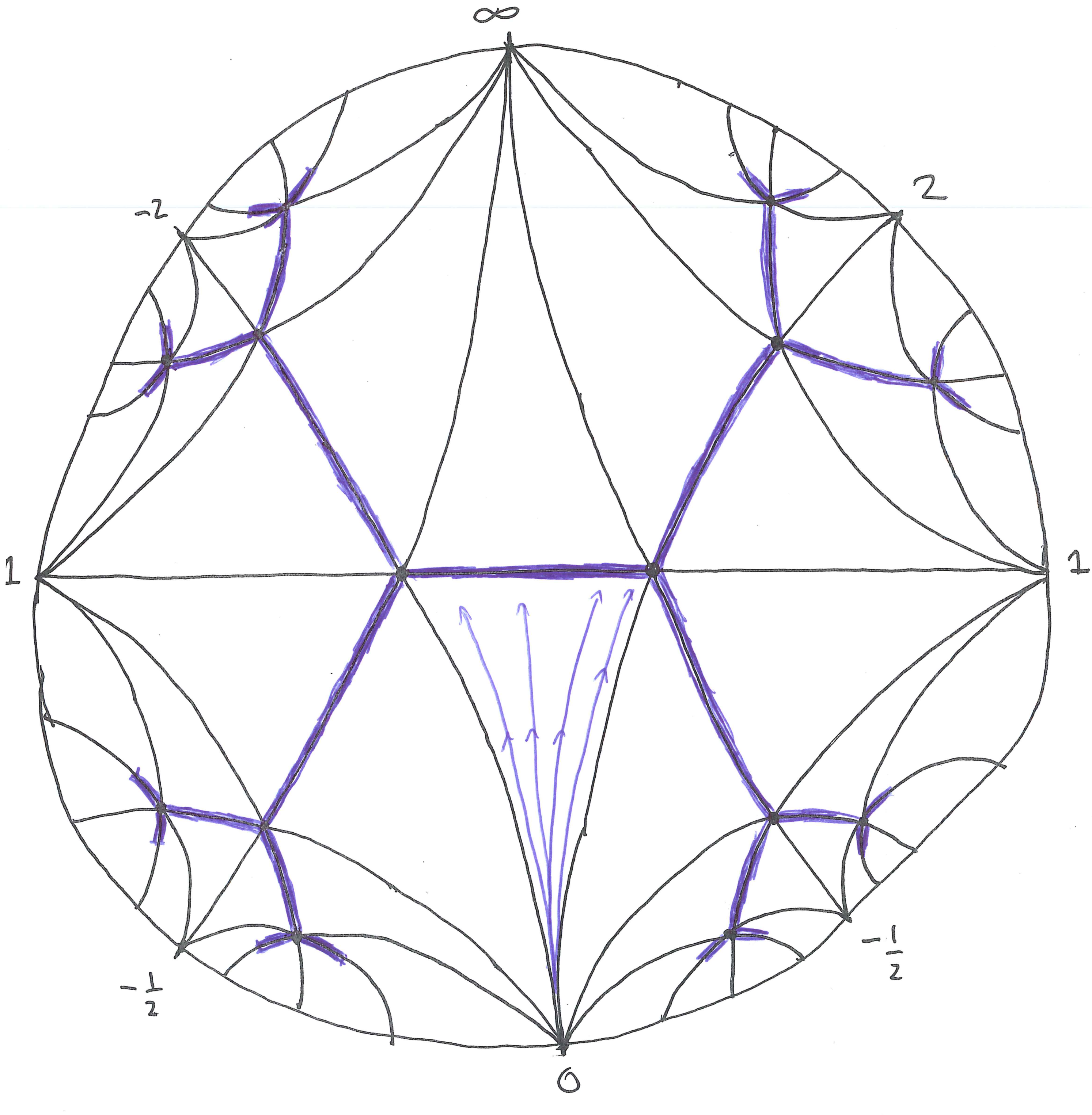
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

① This action leaves invariant a tesselation of $\mathbb{H}^2$ pictured in FIG 1 (below).

② As in FIG 1, there is a deformation retraction of the fundamental domain onto a boundary edge.

③ Considering the $SL_2\mathbb{Z}$ action, this induces an $SL_2(\mathbb{Z})$-equivariant deformation retraction of $\mathbb{H}^2$ <u>onto a tree</u>. (in blue)

FIG 1



Conclusion, $\boxed{SL_2\mathbb{Z} \circlearrowright \text{ tree}}$.

Moreover, this action is

- co compact
- has finite cell stabilizer (via a direct computation)

Thus Brown's criterion applies and

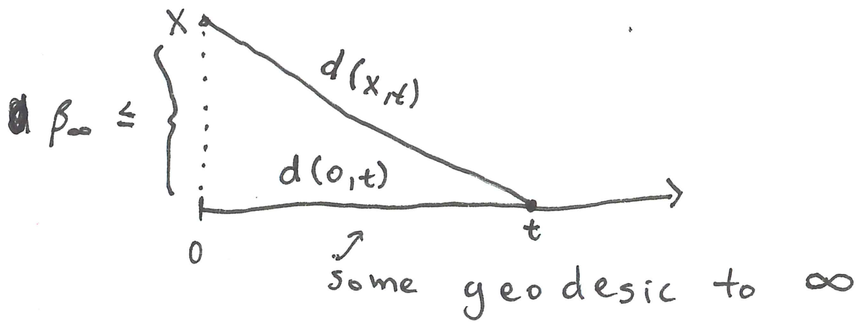
$SL_2(\mathbb{Z})$ is of type $F_\infty$
(i.e. of type $F_m$ for all $m$)

Fact: $SL_2(\mathbb{Z})$ has a torsion free subgroup of finite index that acts

- freely
- cocompactly

on a locally finite tree.

Morse Theory picture
(see FIG 2 below).
Level sets for the Busemann function $\beta(x)$ are pictured.




$$\beta_\infty(x) = \lim_{t\to\infty} d(0,t) - d(x,t)$$

Upshot: We obtain a $SL_2(\mathbb{Z})$ invariant height function

$$h : \mathbb{H}^2 \longrightarrow \mathbb{R}$$

and can apply Morse theory

In particular,

- Vertices of tree (in black for FIG 2) $\longleftrightarrow$ height function attains it's minimum $m$

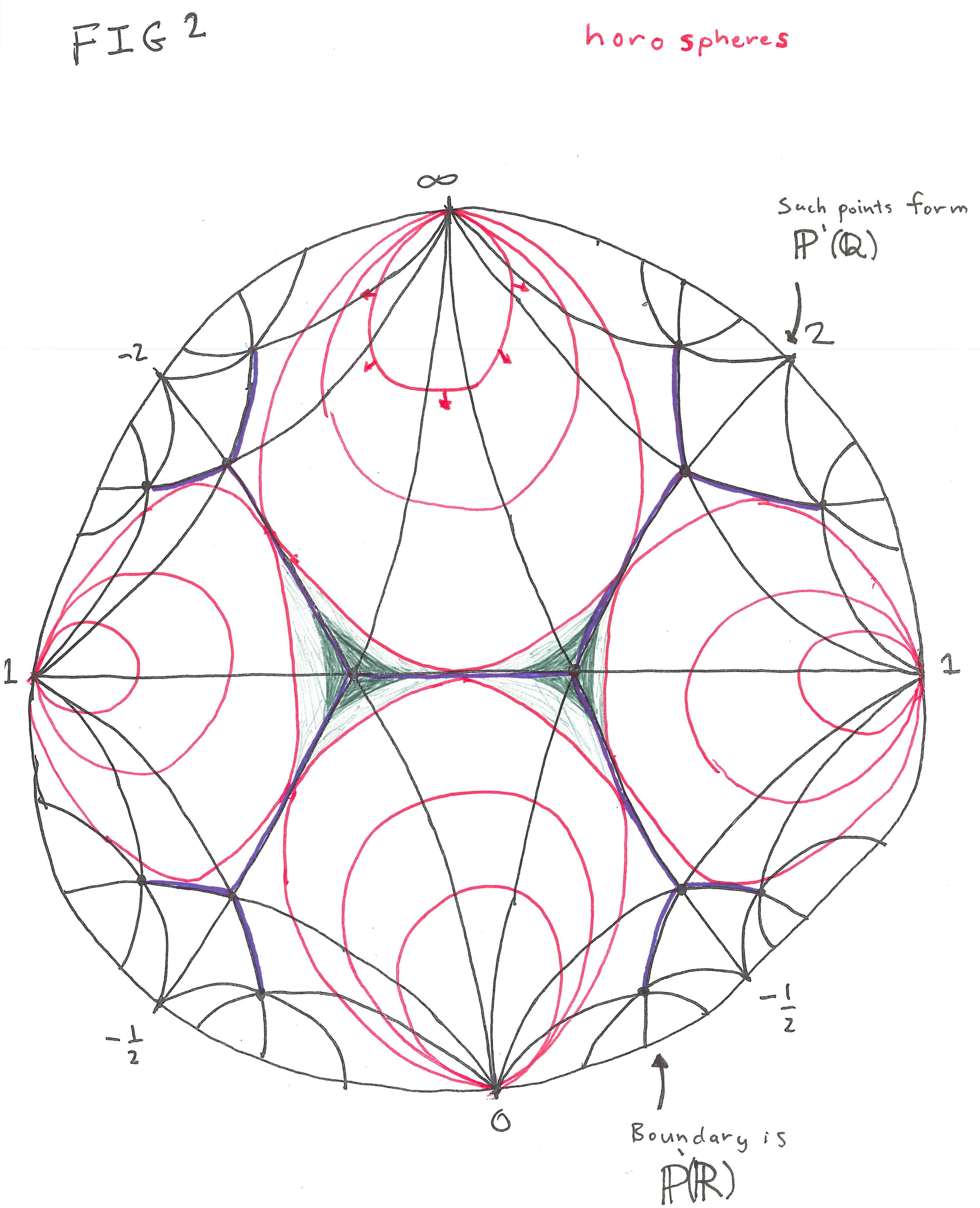
i.e. $h^{-1}((\infty, m]) =$ Set of vertices

- Only critical pts are where horospheres are tangent.

( In FIG 2, ① light + dark green illustrate down sets for this critical point. ② dark green illustrates a down set at a non-critical pt. ③ ~~[illegible]~~ red arrows indicated that later points are not critical )

Conclusion, $\mathbb{H}^2$ is $SL_2(\mathbb{Z})$-equivariantly equivalent to the tree (blue) by means of this height function.

FIG 2


horospheres
Such points form
$\mathbb{P}^1(\mathbb{Q})$
$\infty$
2
-2
1
1
0
$-\frac{1}{2}$
$-\frac{1}{2}$
Boundary is
$\mathbb{P}^1(\mathbb{R})$

For: $SL_2(\mathbb{F}_q[t])$

$\mathbb{F}_q(t)$ : global function field

$\downarrow$

$\mathbb{F}_q((t^{-1}))$ : local function field

$$:= \left\{ \sum_{i \geq n} a_i t^{-i} \right\}$$

$\mathbb{F}_q[t] = \mathcal{O}_{\{deg\}}$ for $\deg : \mathbb{F}_q[t] \longrightarrow \mathbb{Z} \cup \{\infty\}$

$\frac{p(t)}{q(t)} \longmapsto -\deg p + \deg q$

$= \{ f \in \mathbb{F}_q(t) \mid V(f) \geq 0 \; \forall V \neq \deg \}$

Other valuations $V$ are $p$-adic valuations ie.

for $p(t) \in \mathbb{F}_q[t]$ irreducible $\rightsquigarrow$ $p$-adic valuation for $p(t)$.

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$\mathcal{G}(\mathcal{O}_S)$ $\leftarrow$ This is an arithmetic gp.

group scheme

$S$-arithmetic ring

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deg induces a metric on $\mathbb{F}_q(t)$ via

$$a_n \to 0 \iff \deg(a_n) \longrightarrow \infty$$

E.g. $1, t^{-1}, t^{-2}, t^{-3}, \ldots \longrightarrow 0$

In particular, $\mathbb{F}_q(t)$ completed w.r.t. this topology is $\mathbb{F}_q((t^{-1}))$

We wish to reproduce our [redacted] argument for $SL_2(\mathbb{Z})$ so...

__Task__: Find a metric space $X$ s.t.

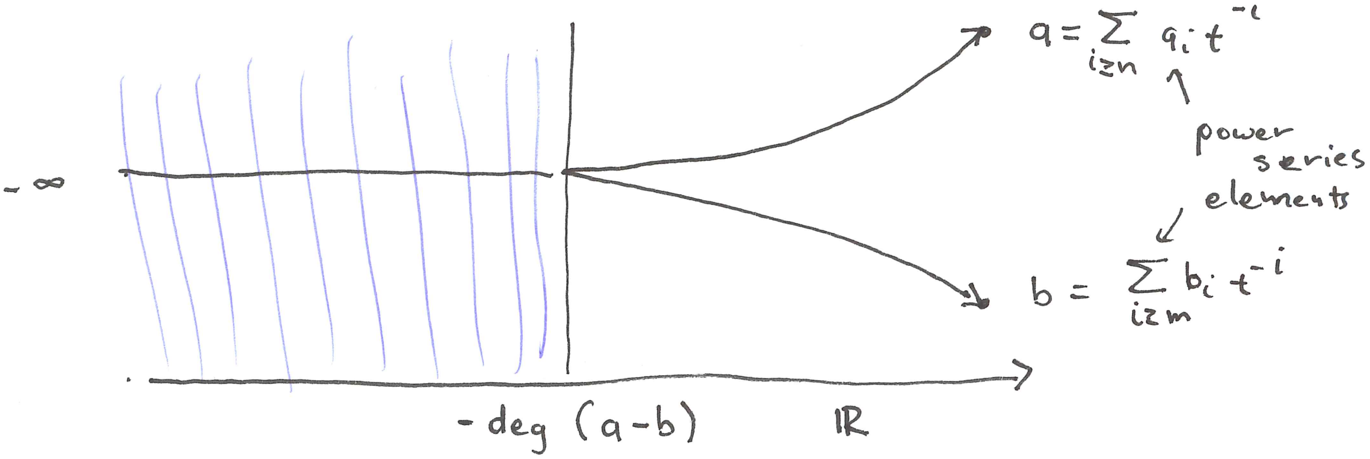
$$\mathbb{P}^1(\mathbb{F}_q((t^{-1}))) = \delta X$$

Probably a tree will work... but we [redacted] need to proform an operation:

ENDS $\rightsquigarrow$ TREE.

We define $\text{TREE} := \coprod_{f \in \mathbb{F}_q((t^{-1}))} (\text{formal geodesics})_f \big/ \text{glueing}$

as in the following picture:

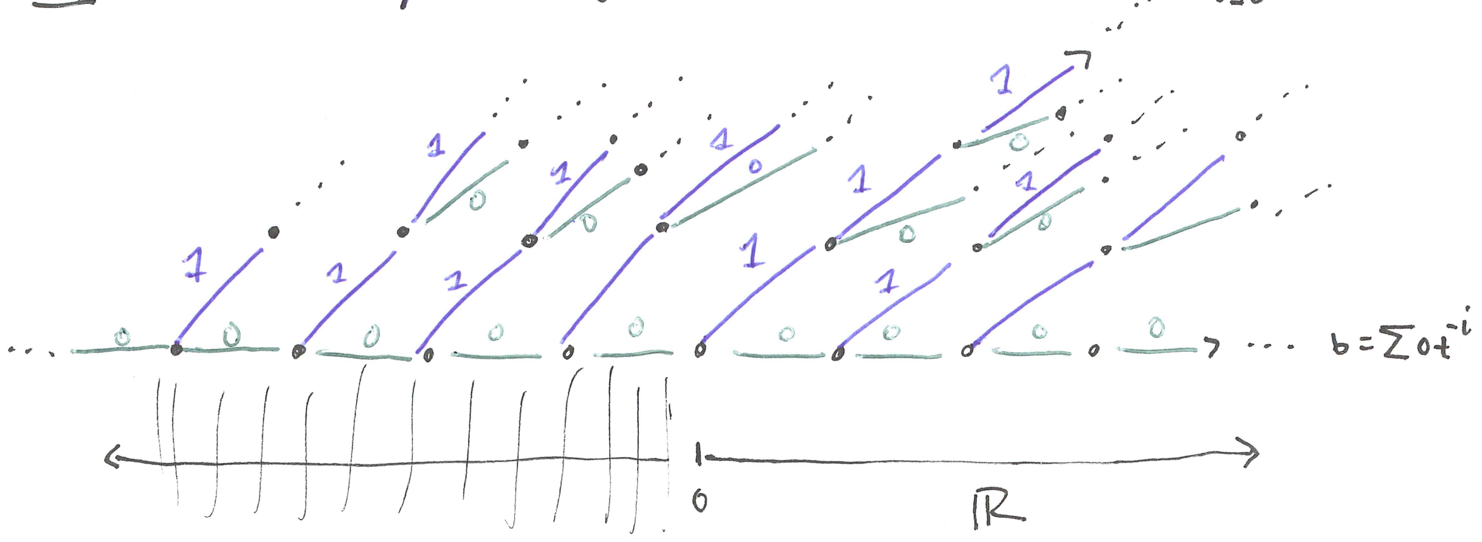



i.e. height of glueing is $-\deg(a-b)$

e.g. $a = \sum_{i \geq 0} 1$ and $b = \sum 0 t^i = 0$

agree on all negetive coefticents and so are glued up to height 0.

Ex: $\mathbb{F}_2 = \{0, 1\}$ gives the tree




$SL_2(\mathbb{F}_q((t^{-1}))) \circlearrowright X$ since

clearly $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \circlearrowright X$ $\xrightarrow[\text{to}]{\text{extends}}$ $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \circlearrowright X$, etc...

Fact: $SL_2(\mathbb{F}_q((t^{-1}))) \circlearrowright X$ s.t.

$SL_2(\mathbb{F}_q((t^{-1}))) \circlearrowright \mathbb{P}[\mathbb{F}_q((t^{-1}))] = \partial X$

Next: generalize the Morse theory picture

Fact (reduction thy twin trees) $\exists$ height function $h$ $X \xrightarrow{h} \mathbb{R}$ s.t.

① Vertices $\longmapsto \mathbb{Z}_{\geq 0}$

② a) $\overset{u}{\bullet}\overset{e}{\text{—}}\overset{v}{\bullet}$ $\Rightarrow$ $h(v) = h(u) \pm 1$

b) $h(u) > 0 \Rightarrow \exists!$ neighbor $v$ s.t. $h(v) = h(u) + 1$

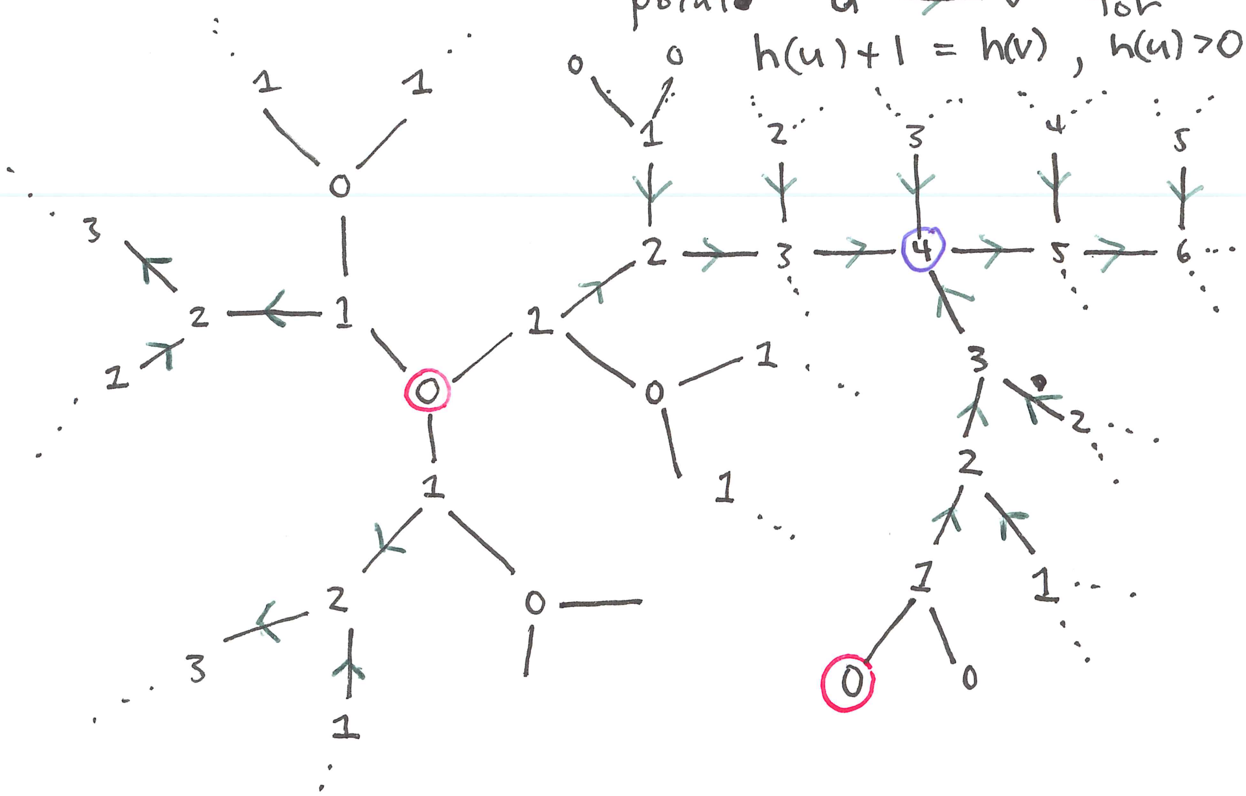
③ $h$ is invariant under the $SL_2(\mathbb{F}_q[t])$-action

Rmk: Notice that $h(u) > 0 \Rightarrow$ $u$ uniquely determines an end:

(2) c) These ends are rational i.e. elements of $\mathbb{P}^1(\mathbb{F}_q(t)) \in \partial X$.

Height picture:

green arrows point $u \to v$ for $h(u)+1 = h(v)$, $h(u) > 0$



Cor: $SL_2(\mathbb{F}[t])$ is not finitely generated.

Pf: Set $X_r = h^{-1}[0, r]$ to filter $X$. Then, this filtration is co compact and all hypotheses of Brown's criterion check.

Consider $X_0$. $\pi_0(X_0 \subseteq X_r)$ is not trivial for any $r$ since for all $r$ $\exists\, u, v \in X_0$ such that the path from $u$ to $v$ passes through $X_{r+1} - X_r$. See $r = 4$ case above with $u, v$ circled in red $w \in X_4 - X_3$ circled in blue.