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Finiteness Properties of arithmetic groups I

<u>Def</u>: A group G is of type F_m if:

$\exists\ K(G,1)$ with <u>finite</u> m-skeleton

↑ unique, up to homotopy equiv.

<u>Obs</u>: G is of type $F_m \iff$

G acts on a contractible CW-complex X <u>freely</u> s.t.

$X^{(m)}$ is finite mod G i.e. X/G is compact.

<u>Criterion</u> (Brown)

Suppose $G \circlearrowright X$ ← contractible CW-complex via cell permuting homeos s.t.

① For each cell $e \in X$, the stabilizer G_e is of type $F_{m-\dim(e)}$

(1a G_e fixes e pointwise i.e. we have a rigid action)

NB: Brown does not require ↗.

② There is a filtration of X by G cocompact subcomplexes $(X_\alpha)_{\alpha \in D}$: D is a directed set

(think e.g. $X_1 \subseteq X_2 \cdots \subseteq X = \bigcup_{i \geq 1} X_i$)

Then: G is of type $F_m \iff$

$(\pi_i X_\alpha)_{\alpha \in D}$ is essentially trivial $\forall i < m$

Def Let $(A_\alpha)_{\alpha \in D}$ be a directed system of groups.

(think e.g. $A_1 \to A_2 \to \cdots$)

It is essentially trivial if $\alpha \in D$ $\Rightarrow \exists \beta$ with $\alpha \leq \beta$ and $A_\alpha \to A_\beta$ trivial.

Rmk: $(A_\alpha)_{\alpha \in D}$ essentially trivial $\Longrightarrow$

$$\operatorname{colim}_{\alpha \in D} A_\alpha = 0$$

Obs: If A_α are all finitely generated then the converse of this holds (using that D is directed).

Rmk $\operatorname{Colim}_{\alpha \in D} (\pi_i (X_\alpha))$
$= \pi_i(X) = 0$ (e.g. consider hocolim)

Rmk. When applying Brown's criterion, one will often have:
finite cell stabilizers.

Ex: (Bieri - Stallings)

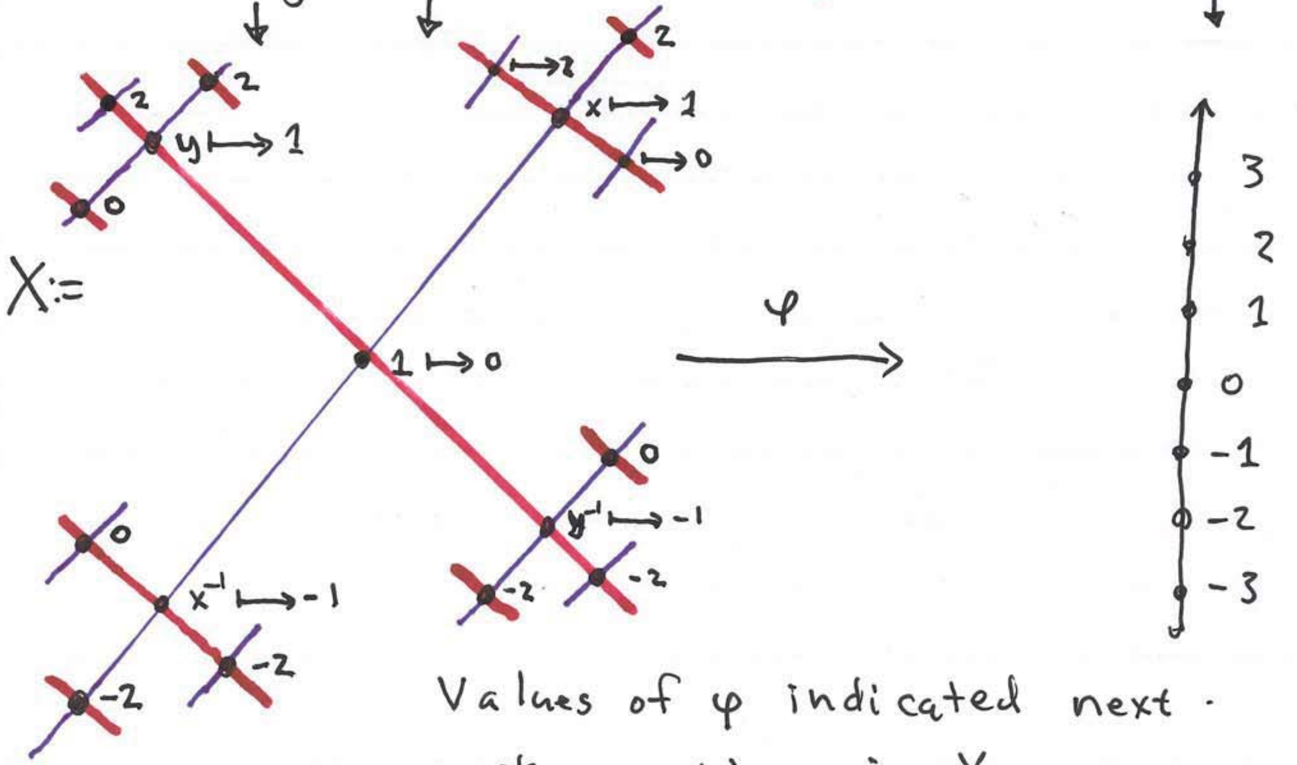
$$H_n := \operatorname{Ker}\left(\langle x_1, y_1 \rangle \times \cdots \times \langle x_n, y_n \rangle \xrightarrow{\varphi} \mathbb{Z} \right)$$

each $\simeq \mathbb{Z} * \mathbb{Z}$ $\qquad x_i, y_i \overset{\varphi}{\longmapsto} 1 \in \mathbb{Z}$

Goal: H_n is of type F_{n-1} but not F_n

n=1 i.e. $H_1 = \operatorname{Ker}(\langle x, y \rangle \xrightarrow{\varphi} \mathbb{Z})$

Cayley graph of $F_2 = \langle x, y \rangle$ $\qquad \mathbb{R}$



Values of φ indicated next to vertices in X.

Suppose $H_1 = \langle \gamma_1, \gamma_2, \cdots, \gamma_n \rangle$

Then <u>filter X</u>: $X_0 \subseteq X_1 \subseteq \cdots \subseteq X = \bigcup_{i \geq 0} X_i$

where $X_r := \text{span}\{ v \mid |\varphi(v)| \leq r \}$

$H_1 \leq F_2 \circlearrowright X$ level preservingly

i.e. $\varphi(\gamma v) = \varphi(v) \quad \forall \gamma \in H_1$

<u>Obs</u> • X_r / H_1 = im (r-ball around 1) is compact

• cell stabilizers are <u>trivial</u>

<u>Conclusion</u>, Brown's Criterion applies and

H_1 is finitely generated ($\Leftrightarrow$ is of type F_1)

$\Leftrightarrow$

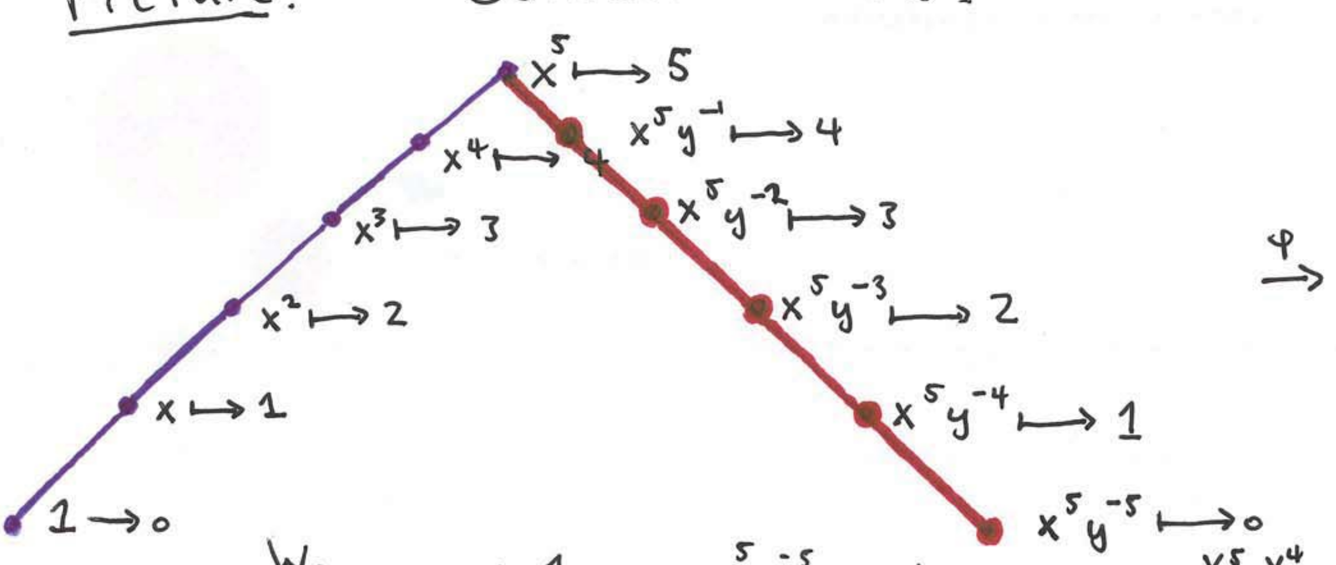
$[\pi_0(X_r)_{r \in \mathbb{N}}]$ is <u>essentially trivial</u>

<u>Question</u>: Does X_0 become connected in some X_r (i.e. is $\pi_0(X_0 \subseteq X_r)$ trivial for some r) ?

No; For any $r \in \mathbb{N}$, there exist vertices v and $w \in X_0$ s.t.

v and w are in <u>different</u> components of X_r

Picture: Consider $r < 5$.

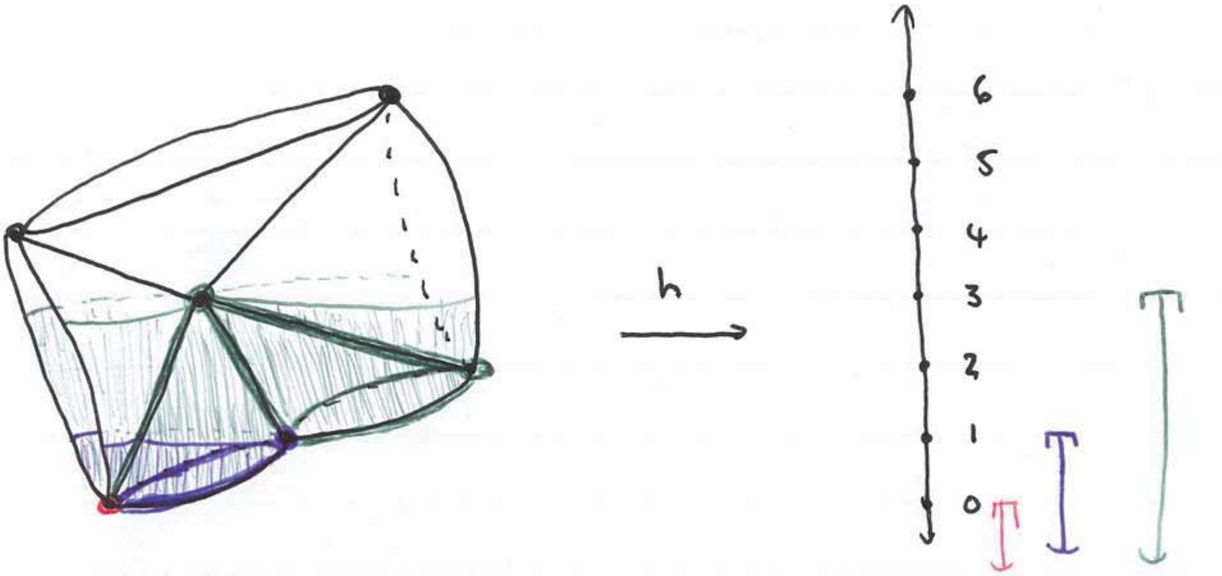


We see $v=1$, we $x^5 y^{-5}$ works, since $x^5 \in X^5 - X^4$.

Conclusion H_1 is not finitely generated.

(Combinatorial) Morse theory

Consider the union of [redacted] cones:



with h the height, as drawn. This is a (classical) Morse function. Above we have (lightly shaded) downsets: $h^{-1}((-\infty, 0])$, $h^{-1}((-\infty, 1])$, $h^{-1}((-\infty, 3])$

But we also see that each of these down sets deformation retracts onto a darkly shaded subcomplex:

The point of combinatorial Morse theory is that:

Vertices ⟷ critical points (in the classical analogy)

so that the:

homotopy type (possibly) changes by coning off the <u>descending link</u>

(contractible [redacted] descending link ⇒ no change)

and "down sets" are then subcomplexes,

as above, that inductively build the original complex.