Proper finiteness properties of certain solvable arithmetic groups

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Classifying spaces, loop spaces and finiteness
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Outline

Actions with prescribed stabilizers

Let a group G act on a CW-complex X (always assume that G_σ fixes $\sigma \in X$ pointwise). Then X is a G-CW-complex.
Let $\mathcal{F}$ be a family of subgroups of G:

- $H \in \mathcal{F} \implies H^g \in \mathcal{F}, g \in G$,
- $H \in \mathcal{F}, K < H \implies K \in \mathcal{F}$.

We say that X **has stabilizers in $\mathcal{F}$** if $G_\sigma \in \mathcal{F}$ for every $\sigma \in X$.

Example

- $\mathcal{F} = \{1\} \rightsquigarrow$ free actions.
- $\mathcal{F} = \{\text{finite subgroups}\} \rightsquigarrow$ proper actions.
- $\mathcal{F} = \{\text{virtually cyclic subgroups}\} \rightsquigarrow \ldots$

Classifying spaces for families of stabilizers

The homotopy category of G-CW-complexes with stabilizers in $\mathcal{F}$ has a terminal object $E_{\mathcal{F}}G$.
A model X for $E_{\mathcal{F}}G$ is a **classifying space for G with stabilizers in $\mathcal{F}$** and is characterized by

1. $X^H \cong$ pt. for $H \in \mathcal{F}$,
2. $X^H = \emptyset$ for $H < G, H \notin \mathcal{F}$.

Example

- $\mathcal{F} = \{1\}$: $E_{\mathcal{F}}G = EG$ classifying space for free actions.
  (1) $\Leftrightarrow X \cong$ pt., (2) $\Leftrightarrow$ free action.
- $\mathcal{F} = \{$finite subgroups$\}$:
  $E_{\mathcal{F}}G = \underline{E}G$ classifying space for proper actions
  (e.g. G acting properly on a CAT(0)-cell complex X).

Finiteness properties

G is **of type $\mathcal{F}$-F_n** if there is a model for $E_{\mathcal{F}}G$ whose n-skeleton is finite modulo the action of G.

Again set $F_n := \{1\}$-F_n and $\underline{F}_n := \{\text{finites}\}$-$F_n$...

Example

1. G is $F_1 \Leftrightarrow G$ is finitely generated.
2. G is $F_2 \Leftrightarrow G$ is finitely presented.
3. G is $\underline{F}_0 \Leftrightarrow G$ has finitely many conjugacy classes of finite subgroups.

Cohomological finiteness properties

G is **of type $\mathcal{F}$-$\boldsymbol{FP_n}$** if there is a projective resolution of the trivial $\mathcal{O}_{\mathcal{F}}G$-module $\underline{\mathbb{Z}}$ that is of finite type up to dimension n.

Facts:

- $\mathcal{F}$-$F_n \implies \mathcal{F}$-FP_n
- $\mathcal{F}$-FP_n and $\mathcal{F}$-$F_2 \implies \mathcal{F}$-F_n (Hurrewicz)

Remarks

- If G is torsion free then $EG = \underline{E}G$.
- $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ is of type F_{n-1} but not of type $\underline{F}_0$.
- Leary–Nucinkis '03: G can be virtually of type F and of type $\underline{F}_{m-1}$ but not $\underline{F}_m$ any m.
- Kochloukova–Martínez-Pérez–Nucinkis '11: groups of type F_{n-1} but not F_n and of type $\underline{F}_{m-1}$ but not $\underline{F}_m$ for some $m \leq n$.

Connectivity

Say that X is

- $\mathcal{F}$**-n-connected** if X^F is n-connected for every $F \in \mathcal{F}$.
- $\mathcal{F}$**-n-acyclic** if $\tilde{H}_k(X^F) = 0$ for every $k \leq n$ and $F \in \mathcal{F}$.

Essential connectedness

A directed system $(X_\alpha)_{\alpha \in D}$ of G-CW complexes is

- $\mathcal{F}$**-essentially n-connected** if

$$\forall \alpha \, \exists \beta \geq \alpha \quad \forall F \in \mathcal{F} \, \forall k \leq n \, \pi_k(X_\alpha^F \to X_\beta^F) = 0$$

- $\mathcal{F}$**-essentially n-acyclic** if

$$\forall \alpha \, \exists \beta \geq \alpha \quad \forall F \in \mathcal{F} \, \forall k \leq n \, \tilde{H}_k(X_\alpha^F \to X_\beta^F) = 0$$

Observation

If $(X_\alpha)_\alpha$ and $(X'_{\alpha'})_{\alpha'}$ are G-cocompact filtrations of a space X then one is essentially n-connected (n-acyclic) if and only if the other one is.

If X is a locally finite affine G-cell complex and $H \subseteq X$ is G-cocompact, say that H is $\mathcal{F}$-essentially n-connected if $(B_n(H))_{n \in \mathbb{N}}$ is.

Brown's criterion

Theorem (Brown '87 $\mathcal{F} = \{1\}$, Fluch–W. '13)

Let G be a group and let $\mathcal{F}$ be a family of subgroups. Assume that G acts on a CW-complex X such that

- *X is $\mathcal{F}$-$(n-1)$-acyclic,*
- *G_σ is of type $(\mathcal{F} \cap G)$-FP_{n-p} for every p-cell of X.*

Let $(X_\alpha)_{\alpha \in D}$ be a cocompact filtration of X. Then G is of type $\mathcal{F}$-FP_n if and only if $(X_\alpha)_\alpha$ is $\mathcal{F}$-essentially $(n-1)$-acyclic.

Proper finiteness properties related to Abels's groups

Theorem (W. '13)

For $0 < m \leq n$ there is a solvable algebraic group $\mathbf{G}$ such that for every prime $p > 2$ the group $G := \mathbf{G}(\mathbb{Z}[1/p])$ is of type F_{n-1} but not of type F_n and is of type $\underline{F}_{m-1}$ but not of type $\underline{F}_m$.

Given $v \in \mathbb{Z}^{n+1}$ subject to some conditions, define

$$\mathbf{G} := \left\{ \begin{pmatrix} d_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & d_{n+1} \end{pmatrix} \in \mathrm{GL}_{n+1} \mid \prod_i d_i^2 = 1 = \prod_i d_i^{v_i} \right\}.$$

Proof outline

$$G := \left\{ \begin{pmatrix} d_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & d_{n+1} \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{Z}[1/p]) \mid \prod_i d_i^2 = 1 = \prod_i d_i^{v_i} \right\}$$

- G acts on a Bruhat–Tits building X.
- Point stabilizers are arithmetic $\implies F_\infty$.
- $\prod_i d_i^{v_i} = 1 \rightsquigarrow G$ leaves a horosphere H invariant.
- The action on H is cocompact.
- Brown '87:
  G is of type $F_n \Leftrightarrow H$ is essentially $(n-1)$-connected.
  Fluch–W. '13:
  G is $\underline{FP}_n \Leftrightarrow H$ is $\{\text{finites}\}$-essentially $(n-1)$-acyclic.

Classical finiteness properties of G

Brown '87: G is of type $F_n \Leftrightarrow H$ is ess. $(n-1)$-connected.

Theorem (Bux–Wortman '11)

A horosphere in an irreducible n-dimensional euclidean building is $(n-2)$-connected.

$\Rightarrow H$ is $(n-2)$-connected

$\Rightarrow G$ is of type F_{n-1} (and not of type F_n).

Proper finiteness properties of G

Fluch–W. '13: G is $\underline{FP}_n \Leftrightarrow H$ is $\{\text{finites}\}$-ess. $(n-1)$-acyclic.
Fact: $H^F = H \cap X^F = H \cap (Y_1 \times \cdots \times Y_k)$

Corollary (to Bux–Wortman '11)

Let X be a euclidean building. A horosphere centered at $\xi \in \partial X$ is $(m-2)$-connected where m is the dimension of the least factor $Y \subseteq X$ such that $\xi \in \partial Y$.

The vector v determines (1) the amount of torsion and (2) the position of the horosphere.

Example

For $v = (m, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{m \text{ terms}})$ the group G is of type $\underline{F}_{m-1}$ but not of type $\underline{F}_m$.