1. Hopkins: Eq Stable Homotopy Theory I

G a finite group. Most things generalize to G compact Lie, but the content of the third lectures don't work as cleanly.

X, Y spaces with a G-action. Study the homotopy classes of G-equivariant maps,

$$[X, Y]^G := \mathrm{Map}^G(X, Y)/\mathrm{htpy}$$

where $\mathrm{Map}^G(X, Y) := \{f : X \to Y | f(gx) = gf(x)\}$.

"Classical" means G is trivial group (i.e. non-equivariantly), we study CW complexes. We then ask "What should a G-CW complex be?"

You learn something if you just linger at this for a minute, since it isn't immediately obvious what this class of spaces should be.

X is a CW complex if it is a space equipped with a collection of (attaching or characteristic) maps $f_\alpha : D^n_\alpha \to X$, with indexing set $\alpha \in S_n$ with $n = 0, 1, 2, \ldots$ such that

- $\coprod_{\alpha \in S_n} \mathrm{int} D^n_\alpha \to X$ is a bijection
- $f(\partial D^n_\alpha) \subset$ finite union of $f(D^m_\beta)$ $m < n$
- $X \to Y$ cts $\iff$ $D^n_\alpha \to X \to Y$ is continuous

The point of putting up this definition is that we have choices about where the group G should act. Clearly it acts on X itself. It could also act on D^n_α or on the indexing set S_n (or both). Want to emphasize that there *is* a decision to be made here. In many ways, the main innovation that we introduced in our work on the Kervaire invariant is the interplay between these two choices.

For us/now, we won't have G act on the disc.

EXAMPLE: $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1 \subset \mathbb{C}$, so G acts by complex conjugation.

- 0-cells = $S^0 = \{2 \text{ elt set with trivial } G - \text{action}\} \times D^0$
- 1-cells = two intervals with nontrivial G action, look like $\{G\} \times D^1$

EXAMPLE: Suppose V is a finite dimensional real representation of G. Since G finite, assume it also has an invariant metric. $S(V)$ =unit sphere in V and S^V is the one point compactification of V. So, when G is trivial and $V = \mathbb{R}^n$ we have that $S(V) = S^{n-1}$ and $S^V = S^n$.

EXAMPLE: $G = \mathbb{Z}/2\mathbb{Z}$ and V=sign representation (underlying vectorspace=$\mathbb{R}$). $S(V) = \pm 1$ and $S^V = S^1$.

Small Theorem: $S(V)$ and S^V admit the structure of G-CW complexes.

Classical Theorems:

(1) X a CW complex, Y a space. If $\pi_n Y = 0$ in every dimension in which X has a cell, *THEN* any map $X \to Y$ is homotopic to a constant map.
(2) **Freundenthal suspension theorem** X CW of dim m, Y $(n-1)-$conn, for $m < 2n$ then $[X, Y] \xrightarrow{\simeq} [S^1 \wedge X, S^1 \wedge Y]$.

The suspension theorem suggests that perhaps we should work stably, since calculating mapping spaces is the same as calculating the suspended version.

Things we'll need G-equivariant analogues of: dimension, connectedness, π_n.

In proving the first theorem, you meet (with S = the set of n-cells, G acting on it):

$$[X^{(n)}/X^{(n-1)}, Y]^G = \prod \pi_n Y$$

where

$$X^{(n)}/X^{(n-1)} = \bigvee_{\alpha \in S} D^n_\alpha / S^{n-1}_\alpha = \bigvee S^n.$$

In equivariant homotopy theory, want to regard $\pi_n Y$ as a functor

$$G-\text{sets} \to \begin{cases} \text{Sets} & \text{n=0} \\ \text{Groups} & \text{n=1} \\ \text{Abelian Groups} & \text{n >1} \end{cases}$$

where the functor sends

$$T \mapsto [\bigvee_{t\in T} S^n, Y]^G = [T_+ \wedge S^n, Y]^G =: \underline{\pi}_n(Y)(T).$$

Where $T_+ = T$ with a disjoint basepoint. Note that $\pi_n^G(Y) = \underline{\pi}_n Y(pt) = [S^n, Y]^G$.

$\underline{\pi}_n Y$ is determined by its value on (coproducts of) transitive G-Sets ($T = (\amalg) G/H$)[1],

$$\underline{\pi}_n Y(T_1 \amalg T_2) = \underline{\pi}_n Y(T_1) \times \underline{\pi}_n Y(T_2)$$

Note that

$$\underline{\pi}_n Y(G/H) = [G/H \wedge S^n, Y]^G = [S^n, Y]^G = [S^n, Y]^H = [S^n, Y^H] = \pi_n Y^H$$

where $Y^H = \{y \in Y | hy = y \ \forall h \in H\}$.

Remembering not just the G-space, but the fixed points. This $\pi_n Y^H$ is really what you need when calculating.

For X a G-CW complex,

$$X = \coprod S_n^{\circlearrowleft G} \times D^n / \sim \text{ and}$$
$$X^H = \coprod S_n^H \times D^n / \sim$$

So X^H is a CW-complex. Now, $\dim^G X$ needs to remember $\dim X^H$ for all $H \subseteq G$.

That is, $\dim^G X$ is a function

$$\{H \subset G\} \to \{0, 1, \ldots\} \qquad \dim^G X(gHg^{-1}) = \dim^G X(H).$$

Similarly for connectivity,

$$C(Y) : \{H \subset G\} \to \{0, 1, 2, \ldots\}$$
$$C(Y)(H) = \text{connectivity of } Y^H$$

Reference[Equivariant Freudenthal Suspension Theorem, in detail]:
Adams "Prerequisites (on equivariant stable homotopy) for Carlssons's lecture"
ALGEBRAIC TOPOLOGY AARHUS 1982 Lecture Notes in Mathematics, 1984, Volume 1051/1984, 483-532, DOI: 10.1007/BFb0075584 `http://www.springerlink.com/content/402616937305u556/`

Theorem 1.1 (Equivariant Freudenthal Suspension Theorem). *If $\dim^G X < K \cdot C^G Y$, then $\forall V$,*

$$[X, Y]^G \to [S^V \wedge X, S^V \wedge Y]^G$$

is a bijection (comment: K is an "equivariant generalization of 2")

1.1. **Equivariant Spanier-Whitehead category.** Denoted SW^G has
- ob: finite G CW complexes
- maps: $\{X, Y\}^G = \text{colim}_V [S^V \wedge X, S^V \wedge Y]^G$ for V representations.

Note: This colim is attained at a finite stage.

Don't have to use *all* representations. Write

$$\rho_G = |G|-\text{dim regular representations of } G = \bigoplus_{g\in G} \mathbb{R} \approx \{f : G \to \mathbb{R}\}.$$

$$\{X, Y\} = \text{colim}_{n\to\infty} [S^{n\rho_G} \wedge X, S^{n\rho_G} Y]^G = [S^{m\rho_G} \wedge X, S^{m\rho_G} Y]^G \ m >> 0.$$

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[1]The transitive G-set G/H comes with a chosen basepoint, H/H, which may cause formulas to look complicated.

Question What is $\{S^0, S^0\}^G$? (i.e. what is the degree of a map).
For this, we need to introduce

Definition 1.2 (Burnside Category of G)**.** *$Burn^G$ has*

objects	*finite G-sets*
morphisms	*correspondences*

What is a correspondence? a class of diagram $S \leftarrow S' \to T$, where two diagrams

$$\begin{array}{c} S \leftarrow S' \to T \\ S \leftarrow S'' \to T \end{array}$$

are in the same equivalence class if we have (for i an iso)

$$\begin{array}{ccccc} & \swarrow & S' & \searrow & \\ & & \downarrow i & & \\ S & \longleftarrow & S'' & \longrightarrow & T \end{array}$$

then $P(S,T)$ is the set of equivalence classes of such diagrams. It is a communative monoid under $\amalg$ in S'

$$\begin{array}{ccccc} & & S'_1 \amalg S'_2 & & \\ & \swarrow & & \searrow & \\ S & & & & T \end{array}$$

then $\mathrm{Burn}^G(S,T)$ is group completion of $P(S,T)$.

Theorem 1.3. *$S \mapsto S_+$ defines a functor $Burn^G \to SW^G$ which is fully faithful.*

CONSEQUENCE: Burnside ring of G=$\mathrm{Burn}^G(\mathrm{pt},\mathrm{pt}) \xrightarrow{\simeq} \{S^0, S^0\}^G$, where $\mathrm{Burn}^G(\mathrm{pt},\mathrm{pt})$ is the Grothendieck group of finite G-sets.

This is illustrating the interplay between the finite G-set point of view and the representation theory point of view.

2. Hopkins: Eq Stable Homotopy Theory II

2.1. **Story:** John Tate's 80'th birthday. Was teaching classical geometry, really excited about these old ideas, standing around the Kummer surface. Told this to John in an excited way. He said "Mike, you're getting old"; when you're young, the math you want to learn is the latest greatest exciting tools and when you get old you get interested in basic things. Was excited by your talks about the latest and greatest things.

2.2. **Recap.** Things: SW^G, X a finite poited $G-CW$-complex,

$$\{X, Y\}^G = \mathrm{colim}_V [S^V \wedge X, S^V \wedge Y]^G = [S^V \wedge X, S^V \wedge Y]^G \; V >> 0$$

also introduced Burn^G, the category with objects finite G-sets and morphisms Burn^G(S,T) the group completion of the set $S \leftarrow U \to T$.

2.3. **Today's Goal.**

Theorem 2.1 (Segal, tom Dieck)**.** *The transformation $S \mapsto S_+$ (adding a basepont) defines a functor $Burn^G \to SW^G$ which is fully faithful.*

For example, this computes

$$\{S^0, S^0\}^G = \text{Burnside ring of } G$$

Today, want to explain the proof of this, because the story of the proof let's me introduce more structure of what's going on in Stable Homotopy theory and also introduce the one really surprising tool.

2.4. **Duality.** First heard about duality in symmetric monoidal categories to talk about Spanier-Whitehead duality.

Suppose C is a symmetric monoidal category with binary op $\otimes : C \times C \to C$ with unit $\mathbf{1}$ with

$$\begin{aligned} (X \otimes Y) \otimes Z &\sim X \otimes (Y \otimes Z) \\ X \otimes \mathbf{1} \sim X &\sim \mathbf{1} \otimes X \\ X \otimes Y &\sim Y \otimes X \end{aligned}$$

Examples:

$$\begin{array}{lcll} C & = & \text{Vector spaces} & \otimes = \otimes \\ C & = & SW^G & \otimes = \wedge \\ C & = & \text{Burn}^G & \otimes = \times \end{array}$$

Definition 2.2. *An object $X \in C$ is **dualizable** if $\exists Y \in C$ and maps $X \otimes Y \to 1$ and $1 \to Y \otimes X$ such that*

$$\begin{array}{ccc} X \otimes 1 & \xrightarrow{\sim} & X \otimes Y \otimes X \\ & \searrow & \downarrow \\ & & 1 \otimes X \end{array}$$

also

$$\begin{array}{ccc} 1 \otimes Y & \xrightarrow{\sim} & Y \otimes X \otimes Y \\ & \searrow & \downarrow \\ & & Y \otimes 1 \end{array}$$

If X is dualizable, then

$$\begin{aligned} C(X \otimes W, Z) &\sim C(W, Y \otimes Z) \\ C(Y \otimes W, Z) &\sim C(W, X \otimes Z) \end{aligned}$$

Ex: $C = Vect$, then dualizable = finite dimensional.

Note: bit misleading. Seems like we should say "X has a dualizable structure..." + all our diagrams with Y. But dualizability is just the existence, you're either dualizable or not.

Consider the functor $W \mapsto C(X \otimes W, 1)$ - representable? (i.e. eq to $C(W, Y)$), if representable, determines Y. Then you form the equations (that follow from X being dualizable)

$$\begin{aligned} C(X \otimes W, Z) &\sim C(W, Y \otimes Z) \\ C(Y \otimes W, Z) &\sim C(W, X \otimes Z) \end{aligned}$$

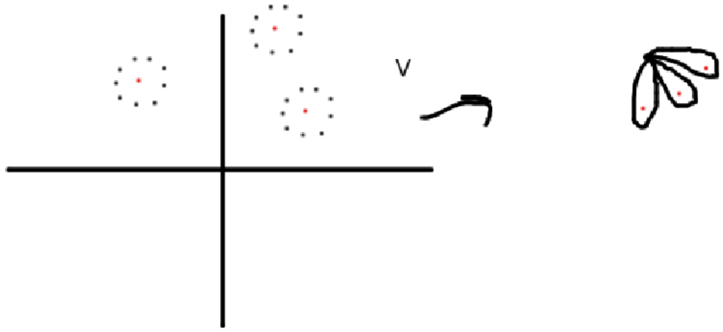
and ask if they're isomorphismss.

Proposition 2.3. *A symmetric monoidal functor sends dualizable objects to dualizable objects.*

$$\begin{array}{rcl} \text{Burn}^G & \to & SW^G \\ T & \to & T_+ \\ S \xleftarrow{f} U \xrightarrow{g} T & \rightsquigarrow & S^V \wedge S_+ \to S^V \wedge T_+ \;\;, V >> 0 \end{array}$$

Choose an equivariant embedding $U \overset{i}{\subset} V$ (Ex $V = \bigoplus_{u \in U} \mathbb{R}$).

This gives me an embedding $U \overset{(i,f)}{\subset} V \times S$
Now do the Pontryagin-Thom collapse.



$$S^V \wedge S_+ \to S^V \wedge U_+ \stackrel{1\wedge g}{\to} S^V \wedge T_+$$

EXERCISE: Check that this is compatible with composition. [This makes you think about and come to grips with this definition. If you want to really absorb this info, you should do this exercise].

This defines $\mathrm{Burn}^G \to SW^G$.
In Burn^G, every object is self-dual

$$T \times T \stackrel{\Delta}{\leftarrow} T \to \mathrm{pt} \in \mathrm{Burn}^G(T \times T, 1)$$

$$\mathrm{pt} \leftarrow T \stackrel{\Delta}{\to} T \times T \in \mathrm{Burn}^G(1, T \times T)$$

EXERCISE: Check that this shows T is its own dual.

Corollary 2.4. *The object $T_+ \in SW^G$ is self-dual (since $\mathrm{Burn}^G \to SW^G$ is symmetric monoidal)*

Philosophizing:
SW^G satisfies:

- stability with respect to representation spheres

$$\{X, Y\} \stackrel{\sim}{\to} \{S^V \wedge X, S^V \wedge Y\}$$

- finite G-sets are self-dual. This says

$$\{S_+, X\}^G = \{S^0, S_+ \wedge X\}$$

If I forget about the group action, then

$$\{S_+, X\} = \prod_{s \in S} \{S^0, X\}$$

or, more suggestively , $\prod_{s\in S} X$.

The other side is $\bigvee_{s \in S} X$. Then this equivalence is saying

$$\prod_{s \in S} X \stackrel{\sim}{\leftarrow} \bigvee_{s \in S} X$$

Recall: In an additive category $\bigoplus \to \prod$ is an iso.
Should be thinking of this equivariant equality as the natural equivariant version of additivity in which the group acts on the indexing set.

THAT IS: Self-duality of finite G-sets is an equivariant version of additivity in which the group G is allowed to act on the indexing set S.

Originally, we emphasized the analogy between G CW-complexes and CW complexes where we let the group act on the cells. Now, duality expresses additivity where the group is allowed to act on the indexing set.

In our paper on the Kervaire Invariant, one of the things we exploited was equivariant multiplication (the norm).

This (self-duality) is an important thing to understand about fnite G-sets.

EXERCISE/point: Remember that SW^G has stability with respect to representations (very topological) and the self-duality (additivity). So,

"Theorem":

$$\frac{\begin{array}{c}\text{Stability with respect to (non-representation spheres) } S^n \\ + \\ \text{self-duality of finite } G\text{-sets}\end{array}}{\iff \text{ stability with respect to } S^V.}$$

Note: There's really two ways you might define stable maps between representation spheres. Could have chosen that the group doesn't act on the spheres involved. In that world, wouldn't've attained "true additivity" because the G-sets would not have been self-dual.

EXERCISE: In SW^G, we have duality maps

$$T_+ \wedge T_+ \to S^0$$
$$S^0 \to T_+ \wedge T_+$$

One of these exists as a map of G CW-complexes. One only exists after smashing with S^V for V suffficiently large.

The question is: which one exists (pre-smashing) and what is the map?

HINT: There's a big clue on the board; natural map always $\bigvee$ to $\prod$.

2.5. **Proof that** $\mathrm{Burn}^G(S,T) \to \{S_+, T_+\}$ **is an iso.**

(1) Use duality to re-write both sides

$$\mathrm{Burn}^G(S \times T, \mathrm{pt}) \to \{S_+ \wedge T_+, S^0\}^G$$

i.e. reduce to the case T =pt.

(2) Both sides take $\amalg$ in the S variable to $\oplus$ [Exercise: prove it].

(3) We reduce to showing $\mathrm{Burn}^G(G/H, \mathrm{pt}) \to \{G/H_+, S^0\}^G$ is an iso.

Let's look at these two sides:

$$\{G/H_+, S^0\}^G = \mathrm{colim}[S^V \wedge G/H_+, S^V]^G = [S^V, S^V]^H$$

Lemma/Exercise: $\mathrm{Burn}^G(G/H, Y) = \mathrm{Burn}^H(\mathrm{pt}, Y)$

[Aside: more generally, we have adjoint functors

$$\mathrm{Burn}^H \rightleftarrows \mathrm{Burn}^G$$
$$S \mapsto G \times_H S]$$

(4) This reduces us to the case $S = T$ =pt,

$$\mathrm{Burn}^G(\mathrm{pt}, \mathrm{pt}) \xrightarrow{\sim} \{S^0, S^0\}^G$$

we'll prove this by induction on $|G|$.

- G=trivial, this is the fact that $\pi_n S^n = \mathbb{Z}$ $n \geq 1$.

Lemma 2.5. $\mathrm{Burn}^G(pt, pt) \to \{S^0, S^0\}^G$ *is a monomorphism, where* $\mathrm{Burn}^G(pt, pt)$ *is the free abelian group on the set of transitive finite G-sets* $\bigoplus_H Z$ *(where H is the conjugacy class of a subgroup)*

For $H \subset G$, $V^H = \{v \in V | hv = v, h \in H\}$

$$\mathbb{Z} = [S^{V^H}, S^{V^H}] \leftarrow [S^V, S^V]^G \to \mathbb{Z}$$

and

$$\begin{array}{ccc} \bigoplus_H \mathbb{Z} = \mathrm{Burn}^G(pt,pt) & \longrightarrow & \{S^0,S^0\}^G \\ & \searrow^{*} & \downarrow \\ & & \prod_H \mathbb{Z} \end{array}$$

(Exercise: Check that $*$ is a mono.)

Proof that $\mathrm{Burn}^G(pt,pt) \to \{S^0,S^0\}^G$ is epi for $G = \mathbb{Z}/2$:

Remember S^σ is a 1-sphere with one 0-cell and a single equivariant 1-cell (where σ is the sign rep of $\mathbb{Z}/2$). There is thus a cofibration sequence

$$\mathbb{Z}/2_+ \to S^0 \to S^\sigma$$

Have map(s)

$$\begin{array}{ccccc} \{S^0,\mathbb{Z}/2_+\}^G & \longrightarrow & \{S^0,S^0\}^G & \longrightarrow & \{S^0,S^\sigma\}^G \\ \uparrow\cong \text{ (by indx)} & & \uparrow \text{ epi?} & & \\ \mathrm{Burn}^G(pt,\mathbb{Z}/2) & \longrightarrow & \mathrm{Burn}^G(pt,pt) & & \end{array}$$

where we "understand"/know $\{S^0,\mathbb{Z}/2_+\}^G \cong \{\mathbb{Z}/,S^0\}^G = \{S^0,S^0\}$ by induction.

We reduce to showing

$$\begin{array}{ccc} \{S^0,S^0\}^G & \longrightarrow & \{S^0,S^\sigma\}^G \\ \uparrow & \nearrow_{\text{epi}} & \\ \mathrm{Burn}^G(pt,pt) & & \end{array}$$

and

$$\{S^0,S^\sigma\}^G = \mathrm{colim}_V[S^V,S^V\wedge S^\sigma]^G$$

with $V_0 \subset V$ the vectors fixed by all $g \in G$.

We'll see next time that the following map is an iso.

$$[S^V,S^V\wedge S^\sigma]^G \to [S^{V_0},S^V\wedge S^\sigma]^G = [S^{V_0},S^{V_0}\wedge S^0] = [S^{V_0},S^{V_0}] = \mathbb{Z}$$

3. Hopkins: Eq Stable Homotopy Theory III

3.1. **preface.** Not going to finish the proof, actually. Decided that there's a lot of things I wouldn't be able to tell you if I did. Might write a little note to add to the notes that will be posted.

3.2. SW^G**.** Introduced the Spanier Whitehead category SW^G. This category doesn't have enough objects. E.g. can't form the mapping cone of a map $x \to y$, since it won't exist (maybe) until smashing with S^V. *Could* form S^V smash the mapping cone, but that might not be equivalent to S^V smash a space.

Solution: For the category of G-Spectra S^G and add a formal desuspension S^{-V}. Can define what you mean as

$$\mathrm{ho}S^G(X,S^{-V}\wedge) = [X,S^{-V}\wedge Y]^G = \{S^V\wedge X,Y\}^G = \mathrm{Map}_{SW^G}(S^V\wedge X,Y)$$

If you make the finite G-sets dualizable and add the finite spheres... (get the finite negative spheres??).

Every object of S^G can be (functorially) written as a filtered colimit

$$\mathrm{colim}_V\, S^{-V}\wedge X_V.$$

In the G-htpy theory of spaces, think of $\pi_n X(-)$: finite G-Sets to abelian groups, taking $\coprod$ to $\oplus$.

In S^G, we have,

$$\pi_n X = [S^n,X]^G$$
$$\text{for T a G-Set}, \pi_n X(T) \quad = \quad [S^n\wedge T_+,X]^G$$

Because of the theorem about

$$\mathrm{Burn}^G(S,T) \xrightarrow{\sim} \{S_+, T_+\}^G = [S_+, T_+]^G,$$

$\pi_n X$ is a contravariant, additive functor

$$\mathrm{Burn}^G \to \text{Abel gps.}$$

Such a functor is called a *MACKEY FUNCTOR*
Question: *Which Mackey functors occur as homotopy groups?*

$$\pi_0 S^0(T) = [T_+, S^0]^G = \mathrm{Burn}^G(T, \mathrm{pt})$$

for S a finite G-set,

$$(\pi_0 S_+)(T) = \mathrm{Burn}^G(T, S).$$

That is, *every representable Mackey functor occurs as a homotopy group*.

$$\begin{array}{rcll} \pi_{-i}S_+ & = & \mathrm{colim}[S^V, S^i \wedge S^V \wedge S_+]^G & i > 0 \\ & = & \mathrm{colim}[G \times_H D^m/S^{m-1}, S^i \wedge S^V \wedge S_+]^G & m \leq \dim V^H \\ & = & [D^m/S^{m-1}, S^i \wedge S^V \wedge S_+]^H & \\ & = & [D^m/S^{m-1}, S^i \wedge S^{V^H} \wedge S_+] = 0 & \text{by conn of spheres} \end{array}$$

Imagine that you give S^V an equivariant cell decomposition, study this colim/maps by equivariant cell decomposition.

Corollary 3.1. *Any Mackey functor occurs as a homotopy group*

$\rightsquigarrow$ EM spectrum HM (for M a Mackey functor) has the property

$$\pi_i HM = \begin{cases} M & i = 0 \\ 0 & \text{else} \end{cases}$$

3.3. **Building abelian groups as homotopy groups.** A an abelian group, want to construct $X$with $\pi_n X = A$.

Starting by writing down a presentation of A by free abelian groups:

$$F_1 \to F_2 - >> A$$

then you form a wedge of spheres

$$\vee S^n \to \vee S^n \to X$$

where $\pi_n \vee S^n = F_1$ and $\pi_n X = A$; this construction depends on knowing $[S^n, S^n]$ and $[S^i, S^n] = 0$ for $i < n$.

This arises in equivariant htpy theory as well.

3.4. **Equivariantly.** in S^G, $[S^n, S^n \wedge T_+]^G$ is a representable (free) Mackey functor and $[S^i, S^n \wedge T_+]^G = 0$ for $i < n$.

Note: won't distinguish notationally between a space and its suspension spectrum. We let X be a pointed G CW-complex, M a Mackey functor,

$$\begin{array}{rcl} H^n_G(X;M) & = & [X, S^n \wedge HM]^G \\ H_n^G(X;M) & = & [S^n, HM \wedge X]^G \end{array}$$

We note that $H^n_G(X;M)$ can be computed as the cohomology groups of the cellular cochains on X with coefficients in M (M is a G-mackey functor, so suppressing the G notation).

$$C^*_{cell}(X;M) = C^0_{cell}(X;M) \to C^1_{cell}(X;M) \to \cdots$$

Note: $C^n(X;M) = M(T_n)$, the set of all n-cells of $X = T_n \times D^n$.

This is, however, a bit off, since so far M is only defined on finite G CW-complexes, and we didn't restrict X to finite. This can be fixed by limiting over the finite subcomplexes.

3.5. **A good example to think through (i.e. EXERCISE).** Let

ρ_G be the regular representation of G
$\bar{\rho}_G = \rho_G$ minus the trivial rep'n
$X = S(\bar{\rho}_G)+$

(Cell decomposition on X): where the unit sphere in a vector space is homeo to the simplex. This unit sphere X, is $\partial\Delta[G]$, the boundary of the standard simplex with vertices in G.

3.6. **Moving on after we have EM Spaces - Postnikov towers.** That is, there is an equivariant Postnikov tower.

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ X & \longrightarrow & \widetilde{P}^n X \\ & & \downarrow \\ & & \widetilde{P}^n X \\ & & \downarrow \\ & & \vdots \end{array}$$

with $\pi_i X \to \pi_i \widetilde{P}^n X$ iso for $i \leq n$ and $\pi_i \widetilde{P}^n X = 0$ for $i > n$. Also , for $M = \pi_n X$

$$S^n \wedge HM = \widetilde{P}^n_N X \to \widetilde{P}^n X \to \widetilde{P}^{n-1} X$$

In classical homotopy theory, an ideal situation is when

$$\widetilde{P}^n_n X = S^n HF$$

for F a free abelian group.

You *might* think that in equivariant homotopy theory, the ideal situation is when $\widetilde{P}^n_n X = S^n \wedge HM$, for M a free Mackey functor. That is, that

$$M(S) = \bigoplus_j \mathrm{Burn}^G(S, T_j)$$

in other words, that $M = \pi_0 T_+$ for T some G-set.

This doesn't really come up in any/many of the naturally occurring geometric examples.

3.7. **Example: Atiyah's Real K-Theory $K\mathbb{R}$.** The $\mathbb{R}$ is a bit misleding. This is what you'd get if you thought of K theory as defined over the real numbers and studying the complex points.

This is a $\mathbb{Z}/2$-equivariant cohomology theory.

$$K\mathbb{R}^0(X) = \text{Grothendieck group of real vector bundles on } X$$

where a *real vector bundle* is a complex vector bundle over X, $V \to X$, equipped with a $\mathbb{Z}/2$ action (τ our generator of $\mathbb{Z}/2$)

$$\begin{array}{ccc} V & \xrightarrow{\tau} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & X \end{array}$$

where τ is conjugate linear

$$\tau(\lambda v) = \bar{\lambda}\tau(v)$$

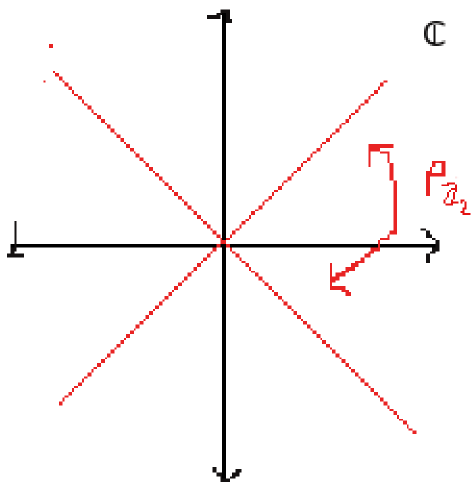
and $\lambda \in \mathbb{C}$.

3.8. Properties of $K\mathbb{R}$:

(1) $X = Y \times \mathbb{Z}/2$, where Y has a trivial $\mathbb{Z}/2$ action. In this case, $K\mathbb{R}(X) = K(Y)$ [that is, you can recover 'ordinary' complex K-theory]
(2) If X has trivial $\mathbb{Z}/2$-action, $K\mathbb{R}(X) = KO(X)$ Grothendieck group of real X-vector bundles. This really arises naturally from geometry.
(3) Periodicity. Atiyah observed that the classic Bott periodicity theorem applies to $K\mathbb{R}$-theory.

$$K\mathbb{R}(X) \sim K\mathbb{R}(S^{\mathbb{C}} \wedge X)$$

where $S^{\mathbb{C}}$= one point compatification of $\mathbb{C}$ with $\mathbb{Z}/2$ acting by conjugation.
We'd like, however, to rewrite it in hopes of generalizing to other groups. If pick a basis for $\mathbb{C}$ that's the diagonals instead the axes, get that the action of $\mathbb{Z}/2$ is $\rho_{\mathbb{Z}/2}$ and $K\mathbb{R}(S^{\rho_{\mathbb{Z}/2}} \wedge X)$.



3.9. Classical K theory. satisfies...

$$K^0(S^0) = \mathbb{Z} \qquad K^i(X) = K^i(S^2 \wedge X)$$
$$S^2 \wedge K \sim K \qquad \pi_0 K = \mathbb{Z}$$
$$P_0^0 K = H\mathbb{Z} \qquad P_{2n}^{2n} K = S^{2n} \wedge H\mathbb{Z}$$
$$P_{2n+1}^{2n+1} K = 0$$

That is, K has a filtration whose associated graded is

$$\bigvee_{n \in \mathbb{Z}} S^{2n} \wedge H\mathbb{Z}$$

3.10. $K\mathbb{R}$ case. Need to consider the constant Mackey functor $\underline{\mathbb{Z}}$ with

$$\underline{\mathbb{Z}}(S) = \mathrm{Map}^G(S, \mathbb{Z}) = \mathrm{Map}(S/G, \mathbb{Z})$$

which sends

$$S \hookleftarrow^{p} U \to T$$

to

$$\underline{\mathbb{Z}}(S) \to \underline{\mathbb{Z}}(U) \leftarrow \underline{\mathbb{Z}}(T)$$

where $f : U \to \mathbb{Z} \rightsquigarrow f' : S \to \mathbb{Z}$ and $f'(s) = \sum_{x \in p^{-1}(s)} f(x)$. Then

$$\pi_0 K\mathbb{R} = \mathbb{Z}$$

$$\widetilde{P}^0_0 K\mathbb{R} = H\underline{\mathbb{Z}}$$

3.11. **Slice tower.** $S^{\rho_{\mathbb{Z}/2}} \wedge K\mathbb{R} \sim K\mathbb{R}$, doesn't tell me about $\widetilde{P}^n_n K\mathbb{R}$. Suggests looking for a filtration whose associated graded is 1r4t3

$$\bigvee_{n \in Z} S^{n\rho_{\mathbb{Z}/2}} \wedge H\underline{\mathbb{Z}}$$

One can alter the definition of the Postnikov tower to form the **Slice Tower** $X \to P^n X$

$$P^n_n X \to P^n X \to P^{n-1} X$$

where

$$P^n_n K\mathbb{R} = \begin{cases} * & n \text{ odd} \\ S^{n\rho_{\mathbb{Z}/2}} \wedge H\underline{\mathbb{Z}} & n = 2m \end{cases}$$

Note: Case $G = \mathbb{Z}/2$ done by Dan Dugger in his thesis, general G in the Kervaire work of Hopkins, Hill, Ravenel.

3.12. **Ideal cases.**

... for Classical Postnikov tower is when F free abelian and

$$P^n_n = S^n \wedge HF$$

... for equivariant Postnikov tower is when F is a free Mackey functor $\pi_0 T_+ = F$

$$\widetilde{P}^n_n = HF$$

... for slice tower is when $(n = m|H|)$

$$P^n_n X = \bigvee G_+ \wedge_H S^{m\rho_H} \wedge H\underline{\mathbb{Z}}$$

in this case, we call X pure.

3.13. **Theorems about this (Hill-Hopkins-Ravenel):**

Theorem 3.2 (HHR)**.** *Many geometrically occurring equivariant cohomology theories are pure.*

Theorem 3.3 ("Gap Theorem" HHR)**.** *If X is pure (and $G \neq \mathbb{Z}/3$), then $\pi_i X(pt)$ (i.e. homotopy groups of the fixed points) satisfies (looking in the range $-4 < i < 0$)*

$$\pi_i X(pt) = \begin{cases} 0 & i=-3,-1 \\ \text{torsion-free} & i=2 \end{cases}$$

Note: means that $\pi_i K\mathbb{R}(\mathrm{pt}) = \pi_i KO$

$$\begin{array}{ccccccccccccc} \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} \\ -4 & & & & 0 & & & & & & & & \end{array}$$

this gap occurs for pure spectra (which are most of them).