7/6/12, 900 Mike Hopkins, Equiv Stable Homotopy Theory

today: <u>won't</u> finish proof from pt II — focus on other aspects instead

$\mathcal{SW}^G$ - fails to have enough objects: no mapping cones Mf for $f: X \to Y$
- form category of G-spectra, $\mathcal{S}^G$; formal desuspension S^{-V}
$[X, S^{-V} \wedge Y]^G = \{S^V \wedge X, Y\}^G$ - can get S^{-V}'s from only S^{-n}'s, too

Every object of $\mathcal{S}^G$ can be written functorially as $\operatorname{colim}_V S^{-V} \wedge X_V$
in G-htpy theory of spaces: $\pi_0 X$: finite G-sets $\to$ Ab
($\amalg$ to $\oplus$)

in G-spectra, $\pi_n X = [S^n, X]^G$; for $T \in G$-set, $\pi_n X(T) = [S^n \wedge T_+, X]^G$
- using $\mathrm{Burn}^G(S,T) \xrightarrow{\cong} \mathcal{SW}^G(S_+, T_+)$, $\pi_n X$ is a contravariant, add. ftr $\mathrm{Burn}^G \to$ Ab
$\rightsquigarrow$ such functors are <u>Mackey</u> functors
Examples of Mackey functors: Which ones occur as $\pi_n X$?
$\pi_0 S^0(T) = [T_+, S^0]^G = \mathrm{Burn}^G(T, *)$
S fin. G-set; $\pi_0 S_+(T) = \mathrm{Burn}^G(T, S)$
- every representable Mackey functor occurs as $\pi_0 X$
$\pi_i S_+$, $i<0$ $= \lim [S^V, S^i \wedge S^V \wedge S_+]^G$; decompose S^V into equiv cells; induct on skeleta
$\lim [G \times_H D^n/S^{n-1}, S^i \wedge S^V \wedge S_+]^G = [D^n/S^{n-1}, S^i \wedge S^V \wedge S_+]^H$
$= [D^n/S^{n-1}, S^i \wedge S^{V^H} \wedge S_+] = 0$ by conn. of spheres

<u>Or</u>: any Mackey functor occurs as a $\pi_i S$;
Construct E-M spectrum HM, M a Mackey functor, so $\pi_i HM = \begin{cases} M, & i=0 \\ 0, & \text{else} \end{cases}$

one point: build abl gp A as $\pi_n X = A$:
- resolve A: $F_1 \to F_2 \to A$, F_i free; $\bigvee S^n \to \bigvee S^n \to X$, $\pi_n X = A$
- uses $[S^n, S^n]$ known, $[S^i, S^n] = 0$, $i<n$; same situation in equiv setting
$\mathcal{S}^G$: $[S^n, S^n \wedge T_+]^G$ - free representable Mackey ftr; $[S^i, S^n \wedge T_+]^G = 0$, $i<n$
(see same args in motivic htpy thy; facts harder to prove)
Homology, cohomology: X a G-CW cx, pointed $\rightsquigarrow$ suspension spectrum (also X)
M Mackey ftr: $H^n_G(X; M) = [X, S^n \wedge HM]^G$, $H^G_n(X, M) = [S^n, HM \wedge X]^G$

$H^n_G(X; M)$: compute as the cohom gps of $C^*_{\text{cell}, G}(X, M)$ — describe?
$C^0_{\text{cell}}(X, M) \to C^1_{\text{cell}}(X, M) \to \cdots$,
$C^n_{\text{cell}}(X, M) = M(T_n)$, set of n-cells of X is $T_n \times D^n$;

what if X not finite? restrict to finite X, or take $\varprojlim$ over finite subsets everything works for Mackey ftrs as it does for abl gps in ordinary case

Example (to work through): ρ_G = reg rep'n of G; $X = S(\bar\rho_G)_+$ - has G-CW decomp already: $= \partial\underbrace{\Delta[G]}$, simplex w/ G-vertices

$\bar\rho_G$ = complement of triv' rep'n

EM spaces $\to$ Postnikov tower; do equivariantly: $X \to \tilde{P}^n X$, $X \to \tilde{P}^{n-1} X$; $\pi_i X \to \pi_i \tilde{P}_n X$ iso, $i<n$; 0, $i>n$.

$\tilde{P}^n_n X \to \tilde{P}^n X \to \tilde{P}^{n-1} X$ fiber seq.

$S^n \wedge HM$; $M = \pi_n X$

Classically: ideal situation is when $\tilde{P}^n_n X = S^n \wedge HF$, F free abl gp — e.g., MU, KU

Equiv setting: might think ideal is when $\tilde{P}^n_n X = S^n \wedge HM$, M free Mackey ftr $(M(S) = \bigoplus_j \mathrm{Burn}^G(S, T_j))$

- but, don't see this structure often

- see what does arise: real K-thy, $K\mathbb{R}$ - $\mathbb{Z}/2$ equiv cohomthy;

$K\mathbb{R}^0(X)$ = Gr gp of real vector bundles $/X$; real v.b is $V \to X$ $\mathbb{C}\times$ + $\mathbb{Z}/2$ action

$$\begin{array}{ccc} V & \xrightarrow{\tau} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & X \end{array}$$

conjugate linear $\tau(\lambda v) = \bar\lambda \tau(v)$ $(\tau: V_X \to V_{\tau(x)})$

Properties: ① $X = Y \times \mathbb{Z}/2$ (trivial action) $\to K\mathbb{R}(X) = K(Y)$

② X has trivial action: conj action gives descent data for real bundle; get $K\mathbb{R}(X) = KO(X)$

③ periodicity for $K\mathbb{R}$ - restate Bott periodicity in alg geo

$K\mathbb{R}(X) \cong K\mathbb{R}(S^{\mathbb{C}} \wedge X)$, $S^{\mathbb{C}}$ = 1-pt cpt of $\mathbb{C}$, $\mathbb{Z}/2$ acts by $\mathbb{C}\times$ conj

- $\otimes$ w/ tautological bundle is $\mathbb{Z}/2$-equiv

write $\mathbb{C} = \dots \circlearrowleft \mathbb{Z}/2 = \rho_G$; can generalize to other G

classical K-thy: $K^0(pt) = \mathbb{Z}$, $K^i(X) = K^{i+2}(X) = K^i(S^2 \wedge X)$ $\rightsquigarrow \pi_0 K = \mathbb{Z}$, $S^2 \wedge K \simeq K$

then $\tilde{P}^0_0 K = H\mathbb{Z}$; $P^{2n}_{2n} K = S^{2n} \wedge H\mathbb{Z}$, $P^{2n+1}_{2n+1} K = 0$

filtration on K so assoc graded is $\bigvee_{n\in\mathbb{Z}} S^{2n} \wedge H\mathbb{Z}$.

what about $K\mathbb{R}$? need $\pi_0 K\mathbb{R}$: constant Mackey ftr $\underline{\mathbb{Z}}$; $\underline{\mathbb{Z}}(S) = \mathrm{Map}^G(S, \mathbb{Z}) = \mathrm{Map}(S/G, \mathbb{Z})$.

$S \leftarrow U \rightarrow T \leftrightarrow \underline{\mathbb{Z}}(S) \leftarrow \underline{\mathbb{Z}}(U) \leftarrow \underline{\mathbb{Z}}(T)$ (Σ on fibers; comp) $\quad f'(s) = \sum_{x \in p^{-1}(s)} f(x)$

$\pi_0 K\mathbb{R} = \underline{\mathbb{Z}} \to \tilde{P}^0_0 K\mathbb{R} = H\underline{\mathbb{Z}}$; but no info on $\tilde{P}^n_n K\mathbb{R}$;

- $S^{\rho_G} \wedge K\mathbb{R} \simeq K\mathbb{R}$; but S^{ρ_G} not ordinary S^2

— filtration w/ assoc graded $\bigvee_{n\in\mathbb{Z}} S^{n\rho_{\mathbb{Z}/2}} \wedge H\underline{\mathbb{Z}}$ - expect from alg geo

can make a 'slice' tower - modified Post. tower $X \to P^n X$; fiber $P^n_n X \to P^n X \to P^{n-1} X$

- $P^n_n K\mathbb{R} = \begin{cases} *, & n \text{ odd} \\ \Sigma^{k\rho_{\mathbb{Z}/2}} \wedge H\underline{\mathbb{Z}}, & n = 2k \end{cases}$ (Dan Dugger, Ph.D. thesis); H-Hill-Ravenel, general G

Ideal case:

Classical Post. tower: $\tilde{P}^n_n = S^n \wedge HF$, F free abl.

Equiv: $\tilde{P}^n_n = HF$, F free Mackey ftr $\pi_0 T_+$

Slice tower: $P^n_n X = \bigvee G \wedge_H S^{m\rho_H} \wedge H\underline{\mathbb{Z}}$, $n = m|H|$ $\leftarrow$ X <u>pure</u> in this case

<u>Thm</u> (HHR): Many geometrically occurring equiv cohom thys are pure.

Easier: <u>Thm</u> (HHR) if X pure (and $G \neq \mathbb{Z}/3$), $\pi_i X(pt) = 0$, $i = -3, -1$; torsion free, $i = -2$ (usually 0)

- gap thm

<u>ex</u>: $\pi_i K\mathbb{R}(pt) = \pi_i KO$: $\begin{array}{ccccccccc} \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z} \\ -4 & & & & 0 & & & & 4 \end{array}$