Mike Hopkins, Equivariant Stable Homotopy Theory II

Last time: $\mathcal{SW}^G$; obj: X finite pointed G-CW cxs
$\{X,Y\}^G = \operatorname{colim}_V [S^V \wedge X, S^V \wedge Y]^G = [S^V \wedge X, S^V \wedge Y]^G$, $V \gg 0$

Burn^G : obj : finite G-sets ; $\mathrm{Burn}^G(S,T)$ = gp completion of the set of $S \leftarrow Y \rightarrow T$ correspondences

Thm (Segal, tom Dieck): The transformation $S \mapsto S_+$ defines a functor $\mathrm{Burn}^G \to \mathcal{SW}^G$, fully faithful.

Ex: This computes $\{S^0, S^0\}^G$ = Burnside ring of G

today: examine the proof of this thm. - will see some tools along the way

Duality: $\mathcal{C}$ a symmetric monoidal cat: tensor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, unit $\mathbb{1}$
- functorial isos $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$; $X \otimes \mathbb{1} \cong X$; $X \otimes Y \cong Y \otimes X$

ex: $(\mathrm{Vect}, \otimes)$; $(\mathcal{SW}^G, \wedge)$; $(\mathrm{Burn}^G, \times)$

an object $X \in \mathcal{C}$ is dualizable if $\exists\, Y \in \mathcal{C}$, maps $\mathbb{1} \to Y \otimes X$, $X \otimes Y \to \mathbb{1}$

so
$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r] \arrow[dr, "\cong"'] & X \otimes Y \otimes X \arrow[d] \\ & \mathbb{1} \otimes X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \otimes Y \arrow[r] \arrow[dr, "\cong"'] & Y \otimes X \otimes Y \arrow[d] \\ & Y \otimes \mathbb{1} \end{tikzcd}$$

then $\mathcal{C}(X \otimes W, Z) \cong \mathcal{C}(W, Y \otimes Z)$; $\mathcal{C}(Y \otimes W, Z) \cong \mathcal{C}(W, X \otimes Z)$

ex: $\mathcal{C} = \mathrm{Vect}$; dualizable = finite dim'l

note: initially seems like there's a choice of Y; really determined up to unique iso/contractible choice
$\mathcal{C}(X \otimes W, \mathbb{1}) \cong \mathcal{C}(W, Y)$ representable; check 'zorro' composites

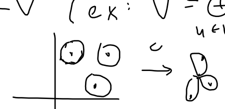
One nice feature of dualizable def'n:

Prop: A symmetric monoidal functor preserves dualizability.

$\mathrm{Burn}^G \to \mathcal{SW}^G$; $T \mapsto T^+$; where to send $S \xleftarrow{f} U \xrightarrow{g} T$?

$\rightsquigarrow S^V \wedge S_+ \to S^V \wedge T_+$, $V \gg 0$ - use Pontryagin-Thom construction

Choose an equivariant embedding $U \overset{i}{\subset} V$ (ex: $V = \bigoplus_{u \in U} \mathbb{R}$, G acts on indices - perm rep'n of U)

gives $U \overset{(i,f)}{\subset} V \times S$; do P-T collapse  $\mapsto S^V \wedge S_+ \to S^V \wedge U_+ \xrightarrow{\mathrm{id} \wedge g_+} S^V \wedge T_+$

point to check: compatibility with composition (forces comprehension of this construction)
- defines $\mathrm{Burn}^G \to \mathcal{SW}^G$ functor

note: in Burn^G, every object is self-dual: $T \times T \xleftarrow{\Delta} T \rightarrow 1$, $1 \leftarrow T \xrightarrow{\Delta} T \times T$; check duality.
- do need to check diagrams. (exercise!)

Cor: T_+ in $\mathcal{SW}^G$ is self-dual, by $T \mapsto T^+$ symm. mon. functor.

2 properties of $\mathcal{SW}^G$: ① satisfies stability: $\{X,Y\} \xrightarrow{\sim} \{S^V \wedge X, S^V \wedge Y\}$ w/r/t rep'n spheres
② finite G-sets are self-dual.
① involves particular spaces - harder to move to other categories - special to topology
② seems more general: $\{S_+, X\}^G = \{S^0, S_+ \wedge X\}^G \Rightarrow \prod_{s\in S} X \xleftarrow{\simeq} \bigvee_{s \in S} X$ - seems additive ($\oplus \simeq \prod$)
- equivariant version of additivity in $\mathcal{SW}^G$
- G allowed to act on index set S.

- parallel to G-action on index sets of cells in G-CW-cxs - exploited in norm maps in HHR
"Thm": Stability w/r/t ordinary spheres + self-duality of finite G-sets is equiv to S^V-stability
Exercise: in $\mathcal{SW}^G$, have duality maps: $T_+ \wedge T_+ \to S^0$, $S^0 \to T_+ \wedge T_+$; one is a map of spaces;
Proof that $\mathrm{Burn}^G(S,T) \xrightarrow{\sim} \{S_+, T_+\}^G$
① use duality to rewrite both sides: $\mathrm{Burn}^G(S \times T, pt) \to \{S_+ \wedge T_+, S^0\}^G$ - reduce to $T = pt$.
both sides take $\amalg$ in S to $\oplus$; reduce to $\mathrm{Burn}^G(G/H, pt) \to \{G/H_+, S^0\}^G$ an iso.
$\{G/H_+, S^0\}^G = \operatorname{colim} \{S^V \wedge G/H_+, S^V\}^G = \{S^V, S^V\}^H$
Lemma/exercise: same true for Burn: $\mathrm{Burn}^G(G/H, Y) \simeq \mathrm{Burn}^H(pt, Y)$ (part of adjunction; $S \mapsto G \times_H S$)
$\to$ reduces to $S = T = pt$: $\mathrm{Burn}^G(pt, pt) \to \{S^0, S^0\}^G$ an iso
induct on $|G|$: G trivial: $\pi_n(S^n) = \mathbb{Z}$, $n \geq 1$.
Lemma: $\mathrm{Burn}^G(pt, pt) \to \{S^0, S^0\}^G$ is injective; $\mathrm{Burn}^G(pt,pt)$ - free ab gp on set of transitive G-sets
$\to \bigoplus_{H/\mathrm{conj}} \mathbb{Z}\langle G/H \rangle$; $H \subset G$; $\{S^V, S^V\}^G \to \mathbb{Z}$ $\xrightarrow{\text{H}}$ $\{S^{V^H}, S^{V^H}\} = \mathbb{Z}$

$$\begin{tikzcd} \mathrm{Burn}^G(*,*) \arrow[r] \arrow[d, equal] & \{S^0, S^0\}^G \arrow[d] \\ \bigoplus_H \mathbb{Z} \arrow[r] & \prod \mathbb{Z} \end{tikzcd}$$

- so check an epimorphism - uses key technique

special case of technique: $G = \mathbb{Z}/2$ first
$= S^\sigma$, sign rep'n - in cofib seq $\mathbb{Z}/2_+ \to S^0 \to S^\sigma$

LES: $\{S^0, \mathbb{Z}/2_+\} \to \{S^0, S^0\}^G \to \{S^0, S^\sigma\}^G$
$\{\mathbb{Z}/2_+, S^0\} \cong \{S^0, \mathbb{Z}/2_+\}$, $\mathrm{Burn}^G(pt, \mathbb{Z}/2) \xrightarrow{\simeq} \{S^0, \mathbb{Z}/2_+\}$, $\mathrm{Burn}^G(pt, \mathbb{Z}/2) \to \mathrm{Burn}^G(pt, pt) \to \{S^0, S^0\}^G$
$\{\mathbb{Z}/2_+, S^0\} = \{S^0, S^0\}$ (induction)
- reduced to showing onto.
$\{S^0, S^\sigma\}^G = \operatorname{colim} \{S^V, S^V \wedge S^\sigma\}^G$; let $V_0 \subset V$ be the invariant space; restrict:
$\{S^V, S^V \wedge S^\sigma\} \to \{S^{V_0}, S^V \wedge S^\sigma\}^G = \{S^{V_0}, \underbrace{S^{V_0} \wedge S^0}_{\text{invt part}}\} = \{S^{V_0}, S^{V_0}\} = \mathbb{Z} \leftarrow$ one $\mathbb{Z}$ factor above.
next time: iso