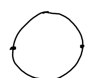
7/2/12, 9:45 am   Mike Hopkins, Equivariant Stable Homotopy Theory I

Goal: explain basics of equiv stable homotopy theory
  - applied in Hill-Hopkins-Ravenel results

Equivariant homotopy theory: G a finite group (much also holds for G compact Lie)
  X, Y spaces, w/ G-action; study $[X, Y]^G = \mathrm{Map}^G(X,Y)/\text{htpy}$
    $\mathrm{Map}^G(X,Y) = \{ f: X \to Y : f(gx) = g f(x) \}$

Classically: Restrict to reasonable spaces: CW complexes
  - what is a G-CW complex? - examine classical ones
  X CW cx: space X, maps $f_\alpha : D^n_\alpha \to X$, $\alpha \in S_n$, $n = 0, 1, 2, \ldots$
      - $\coprod \mathrm{int}\, D^n_\alpha \to X$ is a bijection; $f(D^n_\alpha) \subseteq$ finite union $f(D^m_\beta)$, $m<n$
      - weak topology: $X \to Y$ cts $\Leftrightarrow$ $D^n_\alpha \to X \to Y$ cts
  where could G-action go? on X; D^n_α; index sets S_n
  - Kervaire results depend on D^n, S_n actions, interaction between them
  - for G-CW cx: G action S_n, not D^n_α

Ex: $G = \mathbb{Z}/2$, $X = S^1$,  (unit S^1 in $\mathbb{C}$, action by conjugation
  0-cells: • •, {2 elt set} $\times D^0$
                triv. action
  1-cells: $\{G\} \times D^1$

Ex: Suppose V a finite dim'l real rep'n of G; with metric
  - $S(V)$ = unit sphere in V; S^V = 1-pt cptification of V
  ($G=1$, $V = \mathbb{R}^n$: $S(V) = \mathbb{R}^{n-1}$, $S^V = S^n$)
Ex: $G = \mathbb{Z}/2$, $V = \mathrm{sgn}$ (dim 1); $S(V) = \pm 1$, nontrivial G-action; $S^V = S^1$ (above)
Thm: $S(V)$, S^V admit the structure of G-CW structures
  - view as 'new' building blocks; contrast w/ orig CW definition in pt III

Classical Thms:
1) Suppose X CW complex, Y a space; X has an n-cell $\Rightarrow \pi_n Y = 0$. Then each map $X \to Y$ is homotopic to a constant map.
2) Freudenthal Suspension: X CW cx, dim m; Y $(n-1)$-connected, $m < 2n$: $[X,Y] \xrightarrow{\cong} [S^1 \wedge X, S^1 \wedge Y]$
  need equiv π_n, dim'n, connectivity; then can try to generalize
  - try to write down equivariant proofs

1) use $[X^{(n)}/X^{(n-1)}, Y]$ - $X^{(n)}/X^{(n-1)} = \bigvee_{\alpha \in S_n} D^n/S^{n-1}_\alpha = \bigvee S^n$

  so $[X^{(n)}/X^{(n-1)}, Y] \cong \prod \pi_n Y$
- in equiv HT: $\pi_n Y$ is a functor $G\text{-Sets} \to \begin{cases} \text{sets} & n=0 \\ \text{Gp} & n \geq 1 \\ \text{Ab gps}, & n \geq 2 \end{cases}$, takes $T \in G$-set to $[\bigvee_{t \in T} S^n, Y]^G = [T_+ \wedge S^n, Y]^G$

call $\pi_n Y$ $\underline{\pi}_n Y$; say $\pi_n^G(Y) = \underline{\pi}_n G(pt) = [S^n, Y]^G$

$\underline{\pi}_n Y$ determined by value on finite G-sets - in fact, on transitive G-sets (G/H)

$$\underline{\pi}_n Y(T_1 \amalg T_2) = \underline{\pi}_n Y(T_1) \times \underline{\pi}_n Y(T_2)$$

$$\underline{\pi}_n Y(G/H) = \{G/H_+ \wedge S^n, Y\}^G = [S^n, Y]^H = [S^n, Y^H] = \pi_n(Y^H)$$ H-fixed set

Note: index sets are G-sets; build functor taking G-sets; get to fixed sets later

H/H a distinguished point in G/H; can complicated matters later, especially theoretically

Connectivity + dim'n: fixed points more convenient here

X a G CW cx; $X = \coprod S_n \times D^n/\sim$ (G acting on S_n) $\quad X^H = \coprod S_n^H \times D^n/\sim$ (do need to check both to prove) CW cx

$\dim^G X$ needs to remember $\dim X^H$, $\forall H \subseteq G$

- $\dim^G X$ a function: $\{H < G\} \to \{0, 1, 2, \dots\}$ $\quad \dim^G(gHg^{-1}) = \dim^G(H)$

Similarly, define $C^G(Y): \{H < G\} \to \{0, 1, 2, \dots\}$: $C^G(Y)(H)$ = connectivity of H-fixed pts Y^H.

Equivariant Freudenthal Suspension Thm (see Adams, "Prerequisites for Carlsson's work")

Suppose $\dim^G X < k\, C^G(Y)$. Then $\forall V$, $[X, Y]^G \to [S^V \wedge X, S^V \wedge Y]$ is a bijection

equiv generalization of '2'

Equivariant Spanier Whitehead category: $\mathcal{SW}^G$

obj: finite G CW cxs; mor: $\{X, Y\}^G = \lim_V [S^V \wedge X, S^V \wedge Y]^G$

- classically: limit attained at finite stage; same here;
- use only regular rep'n, ρ_G, to index V's. $\{X, Y\}^G = \lim_{n \to \infty} [S^{n\rho_G} \wedge X, S^{n\rho_G} \wedge Y]^G = [S^{m\rho_G} \wedge X, S^{m\rho_G} \wedge Y]^G$, $m \gg 0$

Q: what is $\{S^0, S^0\}^G$? ($\pi_n(S^n) = \mathbb{Z}$)

Burnside category of G: $Burn^G$

- obj = finite G-sets; mor: correspondences

$$\begin{tikzcd} & S' \arrow[dl] \arrow[dr] \arrow[d, dotted] & \\ S & S'' \arrow[l] \arrow[r] & T \end{tikzcd}$$

$P(S, T)$ = set of equiv classes of correspondence diagrams

- commutative monoid under $\amalg$ in S' slot; $Burn^G(S, T)$ = gp completion of $P(S, T)$ (add inverses)

Thm: $S \mapsto S_+$ defines a functor $Burn^G \to \mathcal{SW}^G$, fully faithful

so $Burn^G(pt, pt) \to \{S^0, S^0\}^G$ an iso;

Burnside ring of G: Grothendieck gp of finite G-sets

- might expect this:

$$\begin{tikzcd} T \arrow[r] \arrow[d] & S^n \arrow[d] \\ X \arrow[r] \arrow[u, bend left] & S^n \end{tikzcd}$$

finite G-set; regular ← can make this work directly; some tricky points

see G/H vs. S^V tradeoff in this thm already.