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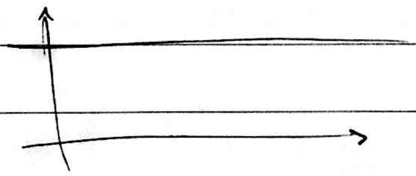
$c \in H^{k+n}(BSO(n))$

$$\begin{matrix} W^n & \longrightarrow & E & & \\ & & \downarrow & \longmapsto & \kappa_c\left(\begin{matrix}E\\\downarrow\\X\end{matrix}\right) \in H^k(X) \\ & & X & & \shortparallel \\ & & & & \pi_!\,(c(T_v E)) \end{matrix}$$

$\pi_! : H^{k+n}(E) \longrightarrow H^k(X)$ using SSS.

E^2



$H^*(X, H^n(W)) = H^*(X, \mathbb{Z})$

$\kappa_c \in H^*(BDiff^+(W))$

<u>$n=1$</u>: $BSO(1) = pt.$

<u>$n=2$</u>: $H^*BSO(2) = \mathbb{Z}[e]$ $\quad |e|=2$

$c \longmapsto \kappa_c$ additive and multiplicative

e^{i+1}

$\kappa_i = \kappa_{e^{i+1}} \in H^{2i}(BDiff^+(W))$ $\qquad$ motivation for this?

for $\mathbb{Q}$-coeff quite a lot is known
$\mathbb{F}$-coeff not so much - $\forall i$:

$$0 = \kappa_i \in H^{2i}(B\mathrm{Diff}^+ W; \mathbb{F}_2)$$

For n=2

Possible W's: $W = \Sigma_g$ genus g surface

Tommasi: (2003) $g=4$

$$H^k(B\mathrm{Diff}^+(\Sigma_4); \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0,2,4,5 \\ 0 & \text{else} \end{cases}$$

as ring $\Vert$

$$\mathbb{Q}[\kappa_1]/\kappa_1^3 \oplus \mathbb{Q}x \qquad |x| = 5$$

$$\kappa_2 = \frac{3}{16} \cdot \kappa_1^2 \qquad \kappa_3 = 0, \ldots$$

Complicated similar to $H^*(B\Sigma_n; \mathbb{F}_2)$

Meaning of $\kappa_1 \neq 0 \in H^2(BDiff(\Sigma_4); \mathbb{Q})$:

$\exists$ $X \xrightarrow{f} BDiff^+(\Sigma_4)$ s.t. $\langle [X], \kappa_1 \rangle \neq 0$

closed oriented

i.e. $\Sigma_4 \longrightarrow E \to X$ $\quad 0 \neq \langle [X], \kappa_1 \begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix} \rangle = \int_X \underbrace{\int_{\text{fibre}} e^2 T_\pi E}_{\pi_!(e^2 T_\pi E)}$

$= \int_E p_1(TE)$

$TE = T_\pi E \oplus \pi^* TX$

$p_1(TE) = p_1(T_\pi E) + \underbrace{\pi^* p_1 TX}_{0}$ $\quad$ $\underset{\text{Atiyah. Sign. thm.}}{=} 3 \cdot \sigma(E)$

Historically this was used the other way to show that $\kappa_1 \neq 0$

$\forall i \; \exists g : \kappa_i \neq 0 \in H^{2i}(BDiff^+ \Sigma_g; \mathbb{Q})$

But $\kappa_i = 0 \in H^{2i}(BDiff^+ \Sigma_g; \mathbb{Q})$ for $i \geq g-1$

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \xrightarrow{(*)} H^*(B\mathrm{Diff}^+ \Sigma_g; \mathbb{Q})$$

neither injective nor surjective. eg.

$g=4$ $\kappa_2 - \frac{3}{16}\kappa_1^2 \in$ kernel

(*)
It is iso for deg ≤ 2

Thm: (Mumford conj. / Madsen-Weiss thm over $\mathbb{Q}$)
(*) is iso in deg $\leq \frac{2(g-1)}{3}$

~~Exam~~ $n=3$: possible $c \in H^*(BSO(3); \mathbb{Q}) = \mathbb{Q}[p_1]$

$|p_1| = 4$

$$\kappa_{p_1^i} \in H^{4i-3}(B\mathrm{Diff}^+(W^3); \mathbb{Q})$$

Thm (Ebert): $\kappa_{p_1^i} = 0$ for all W

and similar(?) for all odd dimensions

$\underline{n=6}$ (4 dangerous)

$$c \in H^*(BSO(6);\mathbb{Q}) = \mathbb{Q}[p_1, e, p_2]$$

$$\rightsquigarrow$$

$$\kappa_{p_1^i e^j p_2^k} \in H^{4i+6j+8k-6}(B\mathrm{Diff}^+ W^6; \mathbb{Q})$$

for

$W_g^6 = \#_g S^3 \times S^3$: $\mathbb{Q}[\kappa_{ijk}] \to H^*(B\mathrm{Diff}^+(W_g^6); \mathbb{Q})$

is iso in deg $\leq \frac{g-4}{2}$

Families of mflds (other kinds than bundles)

$\underline{n=2}$.

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{R}^k \\ {\scriptstyle \pi}\downarrow & & \\ X & & \end{array}$$

Recall: mfld bundle was a smooth proper submersion

π smooth submersion s.t. $(\pi, f): E \to X \times \mathbb{R}^n$ proper

$\pi^{-1}(x)$ is a smooth 2-mfld w/ proper map to $\mathbb{R}^k$

$D_k(X) = \{\text{such } (\pi, f)\}/\sim$

for $k=0$: $D_0(X) \simeq [X, \cancel{B} \coprod_W BDiff^+(W)]$

<u>fact1.</u> D_k is representable for all k

$$D_k(X) \simeq [X, \mathbb{D}_k]$$

defined in terms of submfld's of $\mathbb{R}^\infty$ (can be done for any vectorspace?)

<u>fact2:</u> There are maps $\mathbb{D}_k \longrightarrow \Omega \mathbb{D}_{k+1}$

$$\mathbb{D}_0 \longrightarrow \Omega \mathbb{D}_1 \longrightarrow \Omega^2 \mathbb{D}_2 \longrightarrow \cdots$$

turns out that these maps are w.e. apart from the first.

Get map $\mathbb{D}_0 \longrightarrow \Omega\mathbb{D}_1 \simeq \Omega^k \mathbb{D}_k$ (i.e. into Ω^∞-space)

$$\mathbb{D}_0 = \coprod BDiff^+(W)$$

a) $H^*(\Omega\mathbb{D}_1; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots]$

b) $BDiff^+(\Sigma_g) \longrightarrow \mathbb{D}_0 \longrightarrow \Omega\mathbb{D}_1$
induces iso in H^k for $k \le \frac{2(g-1)}{3}$ (over $\mathbb{Z}$)

classifies

$BDiff^+ W$ (smth) families of oriented closed mflds

D_k (smth) families of mflds with proper map to $\mathbb{R}^k$

gives deloop