Søren Galatius 3/7 · 12

Moduli spaces

analogies. Vector bundles

$$V \to X$$

$$\mathrm{Vect}_n(X) = \left\{ \begin{array}{l} V \to X \\ n\text{-dim'l real vector bundles} \end{array} \right\} \Big/ \mathrm{iso}$$

Fact: $\mathrm{Vect}_n : \mathrm{Ho} \to \mathrm{Sets}$

is representable by CW-cpx. $BO(n)$

$$\mathrm{Vect}_n(X) \longleftrightarrow [X, BO(n)]$$

$$[f^*\gamma] \longmapsfrom [f]$$

$\gamma \to BO(n)$ canonical bundle.

Explicitly

$$BO(n) = \lim_{N\to\infty} Gr_n(\mathbb{R}^N)$$

proof of representability sketch:

Any vector bundle $V \xrightarrow{\pi} X$ admits embedding

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[dr, "\pi"'] & X \times \mathbb{R}^N \arrow[d] \\ & X \end{tikzcd}$$

Then $X \longrightarrow Gr_n(\mathbb{R}^N) = \left\{ \begin{array}{l} W \subset \mathbb{R}^N \\ \text{lin subspace} \end{array} \middle| \; W \cong \mathbb{R}^n \right\}$

$x \longmapsto j(\pi^{-1}(x))$

also $\mathrm{LinEmb}(\mathbb{R}^n, \mathbb{R}^N) / Gl_n(\mathbb{R})$

Example of moduli space is then

$BO(n) = Gr_n(\mathbb{R}^\infty)$ "moduli space of vector bundles"

as it classifies families of bundles

Then $H^*(BO(n))$ = "characteristic classes"

$c \in H^*(BO(n))$ can be evaluated on $V \to X$:

$$c\left(\begin{array}{c} V \\ \downarrow \\ X \end{array}\right) \in H^*(X)$$

Ex.:

$w_i \in H^i(BO(n); \mathbb{F}_2)$ S.-W.

$p_i \in H^{4i}(BO(n); \mathbb{Z})$ Pontryagin

$$H^*(BO(n); \mathbb{F}) = \mathbb{F}_2[w_1, \ldots, w_n]$$

(This last fact is not obvious)

$w_i \neq 0$: look at $H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty)$

Ex.: "Moduli space of finite sets"

A family of finite sets over X is a covering space $E \longrightarrow X$ (i.e. fin. fibres)

$$\mathrm{Cov}_n(X) = \left\{ \begin{array}{l} n\text{-fold coverings spaces} \\ E \xrightarrow{\pi} X \end{array} \right\} \Big/ (\text{homeo}/X)$$

Representable by CW-cpx. $B\Sigma_n$ and there is a natural isomorph. $[X, B\Sigma_n] \leftrightarrow \mathrm{Cov}_n(X)$

Explicitly:

$$B\Sigma_n = \lim_{N \to \infty} \left\{ A \subset \mathbb{R}^N \;\middle|\; \begin{array}{l} A \text{ subset w/} \\ |A| = n \end{array} \right\}$$

also $\mathrm{Inj}(\{1, 2, \ldots, n\}, \mathbb{R}^N) / \Sigma_n$

Idea of proof:

Any covering space $E \xrightarrow{\pi} X$ admits embedding

$$\begin{tikzcd} E \arrow[rr, hook, "j"] \arrow[dr, "\pi"'] & & X \times \mathbb{R}^N \arrow[dl] \\ & X & \end{tikzcd}$$

and

$$\begin{array}{l} X \\ \ni \\ x \longmapsto j(\pi^{-1}(x)) \subset \{x\} \times \mathbb{R}^N = \mathbb{R}^N \end{array}$$

gives $X \longrightarrow B\Sigma_n$

We may again ask for the cohomology

$$H^*(B\Sigma_n) = \left\{ \begin{array}{l} \text{char. classes of} \\ n\text{-fold covering spaces} \end{array} \right\}$$

Fact: $H^*(B\Sigma_n, \mathbb{Q}) = H^*(pt; \mathbb{Q})$

Ex.:

$$H^*(B\Sigma_2;\mathbb{F}_2)=\mathbb{F}_2[x],\quad |x|=1$$

$$\shortparallel$$

$$H^*(B\Sigma_3;\mathbb{F}_2)$$

$$H^*(B\Sigma_4;\mathbb{F}_2)=\mathbb{F}_2[x,y,z]/(xz),\quad |x|=1\quad |y|=2\quad |z|=3$$

No explicit formula for $H^*(B\Sigma_n;\mathbb{F}_2)$ in general.

Note: There is a map $B\Sigma_n \to B\Sigma_{n+1}$

representing
$$\mathrm{Cov}_n(X) \to \mathrm{Cov}_{n+1}(X)$$
$$(E\to X) \longmapsto (E \amalg X \to X)$$

inducing
$$H^*(B\Sigma_{n+1}) \to H^*(B\Sigma_n)$$

which is an isomorphism for $n \gg *$ (Naka

$$\varprojlim H^*(B\Sigma_n;\mathbb{F}_2) = H^*(B\Sigma_\infty;\mathbb{F}_2)$$

is a polynomial ring in explicit classes.

$B\Sigma_n$ is the moduli space of finite sets of cardinality n classifying families of such.

"Moduli space of manifolds" = classifying space of families of manifolds

Let $E \xrightarrow{\pi} X$ be a smooth fibre bundle with closed fibres. i.e. E, X smooth mfld's $\partial E = \partial X = \emptyset$

$\pi : E \to X$ smooth submersion ($D\pi : TE \to TX$ surjective on each fibre)

proper ($\pi^{-1}(\text{compact}) = \text{compact}$)

for $x \in X$: $\pi^{-1}(x) \subset E$ is a smooth compact mfld.

$$\mathrm{Bun}_W(X) = \left\{ \begin{array}{l} E \xrightarrow{\pi} X \text{ smooth} \\ \text{proper submersion} \\ \text{s.t. } \forall x \in X \ \exists \text{ diffeo} \\ \pi^{-1}(x) \simeq W \end{array} \right\} \Big/ \left(\text{diffeo}/X \right)$$

for given closed mfld W

<u>Fact:</u> For each W, Bun_W is a representable functor. There is a space $B\mathrm{Diff}(W)$ and natural isomorphism
$$[X, B\mathrm{Diff}(W)] \longleftrightarrow \mathrm{Bun}_W(X)$$

for smooth X (note that $B\mathrm{Diff}(W)$ is <u>not</u> a mfld.)

<u>Def.:</u>

$$BDiff(W) := \lim_{N\to\infty} \left\{ \begin{array}{c} Q \subseteq \mathbb{R}^N \\ \text{smooth, closed} \end{array} \;\middle|\; \exists \text{ diffeo } Q \approx W \right\}$$

also $\| \quad \mathrm{Emb}(W, \mathbb{R}^N) / \mathrm{Diff}(W)$

classifies families of bundles as above

Idea of sketch of ~~proof~~...

Given $E \xrightarrow{\pi} X$ w/ fibre W, we can pick (whitney emb.)

$$\begin{tikzcd} E \arrow[rr, hook, "j"] \arrow[dr] & & X \times \mathbb{R}^N \arrow[dl] \\ & X & \end{tikzcd}$$

and get

$$\begin{array}{ccc} X & \longrightarrow & BDiff(W) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & j(\pi^{-1}(x)) \end{array}$$

<u>Variation:</u> For W oriented take

orientation preserving

$$\lim_{N\to\infty} \mathrm{Emb}(W, \mathbb{R}^N) / \mathrm{Diff}^+(W) = BDiff^+(W)$$

This classifies bundles $E \xrightarrow{\pi} X$ as before together with orientation of each fibre $\pi^{-1}(x)$ i.e. orientation of the fibrewise tangent bundle

$$\begin{tikzcd} T_\pi E = \mathrm{Ker}\big(D\pi : TE \arrow[rr] \arrow[dr] & & \pi^* TX\big) \arrow[dl] \\ & E & \end{tikzcd}$$

Characteristic classes of manifold bundles

$$H^*(BDiff(W)) = ?$$

<u>Note</u>: $\dim W = 0$ ~~recovers~~ reduces to covering spaces.

$$|W| = n : \quad BDiff(W) \simeq B\Sigma_n$$

<u>Examples</u>: (characteristic classes of manifolds)

For W oriented, we have the generalized MMM classes in $H^*(BDiff^+(W))$

Recall: If W oriented, then $H^*(W) \xrightarrow{\int_W = \langle -, [W] \rangle} \mathbb{Z}$

This has fibrewise version: If $E \xrightarrow{\pi} X$ is an oriented smooth fibre bundle, then we have a similar map

$$H^{k+n}(E) \xrightarrow{\int_W = \pi_!} H^k(X)$$

Given char. class of oriented vector bundles $c \in H^{k+n}(BSO(n))$

$$\begin{array}{ccc} H^{k+n}(E) & \longrightarrow & H^k(X) \\ \rotatebox{90}{$\in$} & & \\ c(T_\pi E) & \longmapsto & \kappa_c\left(\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right) \end{array}$$

and κ_c is a natural transformation

$$Bun_W \longrightarrow H^*(-)$$

There is an universal

$$\kappa_c \in H^k(B\mathrm{Diff}^+(W))$$

for each $c \in H^{k+n}(BSO(n))$

Q: Given W and c, is there a bundle

$E \xrightarrow{\pi} X$ with fibre W and $\kappa_c\left(\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}\right) \neq 0 \in H^k(X)$?

Q: are they algebraicly independent?

Q: Do they generate $H^*(B\mathrm{Diff}\, W)$?