

Moduli spaces

analogies: Vector bundles

$$V \rightarrow X$$

$$\text{Vect}_n(X) = \left\{ \begin{array}{l} V \rightarrow X \\ n\text{-dim'l real vector bundles} \end{array} \right\} / \cong$$

Fact: $\text{Vect}_n : \text{Ho} \rightarrow \text{Sets}$ is representable by CW-cpx. $BO(n)$

$$\text{Vect}_n(X) \leftrightarrow [X, BO(n)]$$

$$[\mathcal{S}^* \mathcal{E}] \longrightarrow [\mathcal{E}]$$

 $\mathcal{E} \longrightarrow BO(n)$ canonical bundle.

Explicitly

$$BO(n) = \varinjlim_N Gr_n(\mathbb{R}^N)$$

proof of representability sketch:

Any vector bundle $V \xrightarrow{\pi} X$ admits embedding

$$\begin{array}{ccc} V & \xrightarrow{j} & X \times \mathbb{R}^N \\ \searrow \pi & & \swarrow \pi \\ & X & \end{array}$$

$$\text{Then } X \longrightarrow \text{Gr}_n(\mathbb{R}^N) = \left\{ \begin{array}{l} W \subset \mathbb{R}^N \\ \text{lin subspace} \end{array} \middle| \begin{array}{l} W \cong \mathbb{R}^n \end{array} \right\}$$

$$x \longmapsto j(\pi^{-1}(x))$$

also
 $\text{LinEmb}(\mathbb{R}^n, \mathbb{R}^N) / \text{GL}_n(\mathbb{R})$

Example of moduli space is then

$\text{BO}(n) = \text{Gr}_n(\mathbb{R}^{2n})$ "moduli space of vector bundles"
 as it classifies families of bundles

Then $H^*(\text{BO}(n)) =$ "characteristic classes"

$c \in H^*(\text{BO}(n))$ can be evaluated on $V \longrightarrow X$:

$$c \left(\begin{array}{c} V \\ L \\ X \end{array} \right) \in H^*(X)$$

Ex.:

$$w_i \in H^i(\text{BO}(n); \mathbb{F}_2) \quad \text{S.-W.}$$

$$p_i \in H^{4i}(\text{BO}(n); \mathbb{Z}) \quad \text{Pontryagin}$$

$$H^*(\text{BO}(n); \mathbb{F}) = \mathbb{F}_2[w_1, \dots, w_n]$$

(This last fact is not obvious)

$w_i \neq 0$: look at $H^*(\mathbb{R}P^{2n} \times \dots \times \mathbb{R}P^{2n})$

Ex.: "Moduli space of finite sets"

A family of finite sets over X is a covering space $E \rightarrow X$ (i.e. pt. fibres)

$$\text{Cov}_n(X) = \left\{ \begin{array}{l} n\text{-fold covering spaces} \\ E \xrightarrow{\pi} X \end{array} \right\} / (\text{homeo}/X)$$

Representable by CW-cpx. $B\Sigma_n$ and there is a natural isomorph. $[X, B\Sigma_n] \cong \text{Cov}_n(X)$

Explicitly:

$$B\Sigma_n = \lim_{N \rightarrow \infty} \left\{ A \subset \mathbb{R}^N \mid \begin{array}{l} A \text{ subset w/} \\ |A| = n \end{array} \right\}$$

also $\text{Inj}(\{1, 2, \dots, n\}, \mathbb{R}^N) / \Sigma_n$

Idea of proof:

Any covering space $E \xrightarrow{\pi} X$ admits embedding

$$\begin{array}{ccc} E & \xrightarrow{j} & X \times \mathbb{R}^N \\ \pi \searrow & & \swarrow \\ & X & \end{array}$$

$$\text{and } \begin{array}{ccc} X & & \\ \cong \downarrow & & \\ X & \xrightarrow{j(\pi^{-1}(x))} & \{x\} \times \mathbb{R}^N = \mathbb{R}^N \end{array}$$

gives $X \rightarrow B\Sigma_n$

We may again ask for the cohomology

$$H^*(B\Sigma_n) = \left\{ \begin{array}{l} \text{char. classes of} \\ n\text{-fold covering spaces} \end{array} \right\}$$

Fact: $H^*(B\Sigma_n, \mathbb{Q}) = H^*(\text{pt}; \mathbb{Q})$

Ex.:

$$H^*(B\Sigma_2; \mathbb{F}_2) = \mathbb{F}_2[x], \quad |x|=1$$

$$\parallel$$

$$H^*(B\Sigma_3; \mathbb{F}_2)$$

$$H^*(B\Sigma_4; \mathbb{F}_2) = \mathbb{F}_2[x, y, z] / (xz), \quad |x|=1 \quad |y|=2 \quad |z|=3$$

No explicit formula for $H^*(B\Sigma_n; \mathbb{F}_2)$ in general.

Note: There is a map $B\Sigma_n \rightarrow B\Sigma_{n+1}$

representing $\text{Cov}_n(X) \rightarrow \text{Cov}_{n+1}(X)$

$$(E \rightarrow X) \longmapsto (E \sqcup X \rightarrow X)$$

inducing

$$H^*(B\Sigma_{n+1}) \rightarrow H^*(B\Sigma_n)$$

which is an isomorphism for $n \gg * \quad (\text{Naka})$

$$\varprojlim H^*(B\Sigma_n; \mathbb{F}_2) = H^*(B\Sigma_\infty; \mathbb{F}_2)$$

is a polynomial ring in explicit classes.

$B\Sigma_n$ is the moduli space of finite sets of cardinality n classifying families of such.

"Moduli space of manifolds" - classifying space of families of manifolds

Let $E \xrightarrow{\pi} X$ be a smooth fibre bundle with closed fibres. i.e. E, X smooth mfld's $\partial E = \partial X = \emptyset$

$\pi: E \rightarrow X$ smooth submersion $\left(\begin{array}{l} D\pi: TE \rightarrow TX \\ \text{surjective on each} \\ \text{fibre} \end{array} \right)$
proper $\left(\pi^{-1}(\text{compact}) = \text{compact} \right)$

for $x \in X: \pi^{-1}(x) \subset E$ is a smooth compact mfld.

$$\text{Bun}_W(X) = \left\{ \begin{array}{l} E \xrightarrow{\pi} X \text{ smooth} \\ \text{proper submersion} \\ \text{st. } \forall x \in X \exists \text{ diffeo} \\ \pi^{-1}(x) \cong W \end{array} \right\} / \left(\text{diffeo.} / X \right)$$

for given ^{closed} mfld W

Fact: For each W , Bun_W is a representable functor. There is a space $\text{BDiff}(W)$ and natural isomorphism $[X, \text{BDiff}(W)] \leftrightarrow \text{Bun}_W(X)$

for smooth X (note that $\text{BDiff}(W)$ is not a mfld.)

Def.: $\lim_{N \rightarrow \infty} \left\{ \begin{array}{l} Q \subseteq \mathbb{R}^N \\ \text{smooth, closed} \end{array} \right\} \left\{ \exists \text{ diffeo } Q \approx W \right\}$

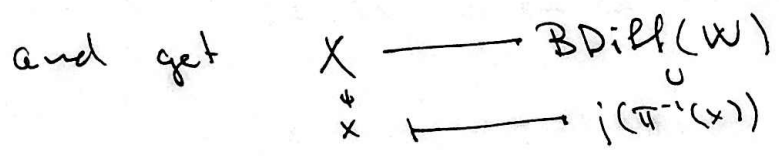
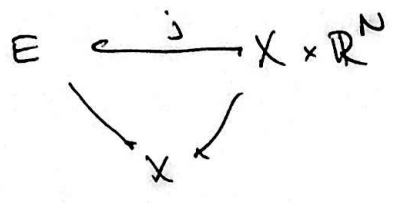
$B\text{Diff}(W)$

also!

classifies families of bundles as above $\text{Emb}(W, \mathbb{R}^N) / \text{Diff}(W)$

Idea of sketch of proof...

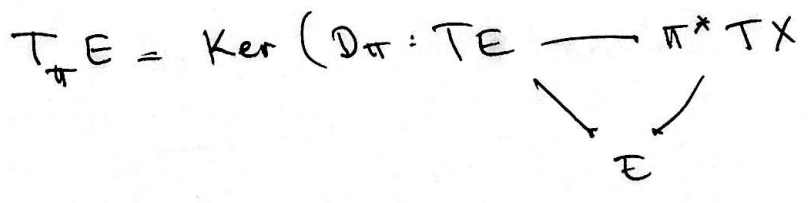
Given $E \xrightarrow{\pi} X$ w/ fibre W , we can pick (whitney emb.)



Variation: For W oriented take orientation preserving

$\lim_{N \rightarrow \infty} \text{Emb}(W, \mathbb{R}^N) / \text{Diff}^+(W) = B\text{Diff}^+(W)$

This classifies bundles $E \xrightarrow{\pi} X$ as before together with orientation of each fibre $\pi^{-1}(x)$ i.e. orientation of the fibrewise tangent bundle



Characteristic classes of manifold bundles

$$H^*(B\text{Diff}(W)) = ?$$

Note: $\dim W = 0$ ~~recovers~~ covering spaces, reduces to

$$|W| = n : B\text{Diff}(W) \cong B\Sigma_n$$

Examples: (characteristic classes of manifolds)

For W oriented, we have the generalized MMM classes in $H^*(B\text{Diff}^+(W))$

Recall: If W oriented, then $H^*(W) \xrightarrow{S_W = \langle -, [W] \rangle} \mathbb{Z}$

This has fibrewise version: If $E \xrightarrow{\pi} X$ is an oriented smooth fibre bundle, then we have a similar map

$$H^{k+n}(E) \xrightarrow{S_W = \pi!} H^k(X)$$

Given char. class of ^{oriented} vector bundles $c \in H^{k+n}(BSO(n))$

$$\begin{array}{ccc} H^{k+n}(E) & \longrightarrow & H^k(X) \\ \downarrow & & \\ c(\pi^*E) & \longleftarrow & \kappa_c \left(\begin{array}{c} E \\ \downarrow \\ X \end{array} \right) \end{array}$$

and κ_c is a natural transformation

$$\text{Bun}_W \longrightarrow H^*(-)$$

There is an universal

$$k_c \in H^k(B\text{Diff}^+(W))$$

for each $c \in H^{k+n}(BSO(n))$

Q: Given W and c , is there a bundle

$$E \xrightarrow{\pi} X \text{ with fibre } W \text{ and } k_c \left(\frac{E}{X} \right) \neq 0 \in H^k(X)?$$

Q: are they algebraically independent?

Q: Do they generate $H^*(B\text{Diff } W)$?