LECTURE 2

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Contents

1. Hochschild chains

Notation 1. *As before, A is an associative algebra over k*

- *Hochschild chains - $C_n(A) = A^{\otimes n+1}$, a reduced variant $A \otimes \overline{A}^n$, $\overline{A} = A/k$*
- *$b : C_n \to C_{n-1}$,*

$$b(a_0, \dots a_n) = \sum_{i=0}^{n-1} a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_n.$$

$(C_*(A), b)$ s a complex computing Hochschild homology of A.

Example 2. *A regular commutative. Set*

$$\Omega^1_{A/k} = \frac{k\text{-span of } adb}{< ad(bc) - cadb - abdc >}$$

$\Omega^1_{A/k}$ is an A-module and we set

$$\Omega^*_{A/k} = \Lambda^* \Omega^1_{A/k}$$

In the case of $A = C^\infty(X)$, we replace $\Omega^1_{A/k}$ by $\Omega^*(X)$. We define the HKR map by

$$I_{HKR} : (C_*(A), b) \to (\Omega^*_{A/k}, 0)$$
$$I_{HKR}(a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 da_2 \ldots da_n$$

I_{HKR} is a morphism of complexes.

Theorem 3. *For A regular commutative algebra, I_{HKR} is a quasiisomorphism.*

Fact 4. *For $A = C^\infty(X)$, the above also holds if we replace $C_n(A)$ by its completion $C^\infty(X^{n+1})$ or alternatively, germs or jets of $C^\infty(X^{n+1})$ on the diagonal $\Delta = \{(x_0, \ldots x_n \mid x_0 = x_1 = \ldots x_n)\}$.*

1.0.1. *Algebraic structure on $(\Lambda^*_A Der(A), \Omega^*_{A/k})$.* **a)** $\Omega^*_{A/k}$ is a differential graded commutative algebra - but the product does not exist when A is not commutative, already $H_0(A, A) = A/[A, A]$ is not an algebra.

b) For a multivector $a \in \Lambda^n_A Der(A)$, there exists a contraction operator

$$\iota_a : \Omega^*_{A/k} \to \Omega^{*-n}_{A/k}$$

In the case of $v \in Der(A)$, ι_v is a degree -1 derivation given by $\iota_v a = 0$, $\iota_v da = v(a)$. We also have the Lie derivative

$$L_a = [d, \iota_a].$$

1.0.2. *Algebraic structures on $(C^*(A, A), C_*(A))$.* **a) The cyclic differential.**

For $C_n(A) = A \otimes \overline{A}$, define $B : C_n(A) \to C_{n+1}(A)$ by

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_i (-1)^{n-i} 1 \otimes a_i \otimes a_{i+1} \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}.$$

Then $B^2 = 0$, $Bb + bB = 0$, hence B induces a differential $H_n(A, A) \to H_{n+1}(A, A)$. Moreover,

$$I_{HKR} : (C_*(A), b) \to (\Omega^*_{A/k}, 0)$$

satisfies

$$dI_{HKR} = I_{HKR} B$$

Hence we can (and will) think of $(C_*(A), b)$ as noncommutative differential forms and of B as de Rham differential.

A good way of keeping track of the two differentials b and B is as follows. Let u be a formal variable of degree -2. Define

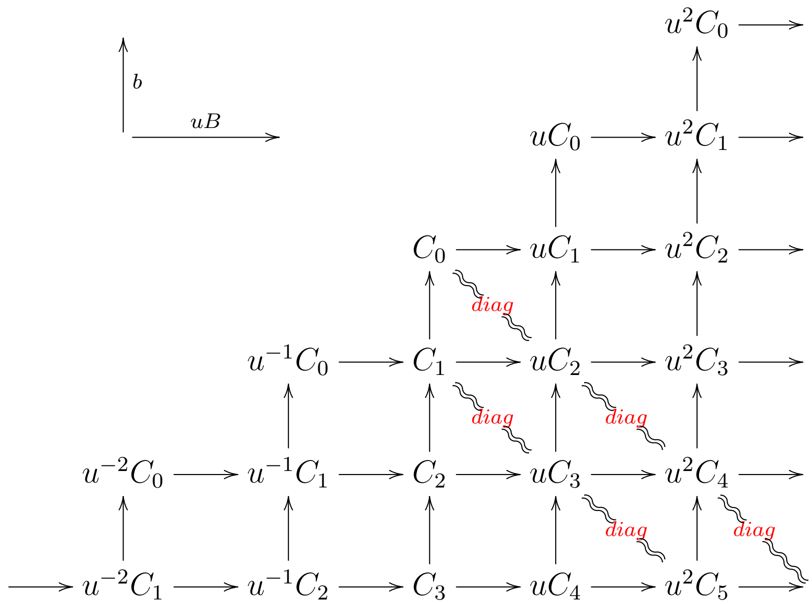
$$CC^-_*(A) = (C_*(A)[[u]], b + uB);$$

$$CC_*^{\mathrm{per}}(A) = \left(C_*(A)\left[\left[u, u^{-1}\right]\right., b + uB\right);$$

$$C_*^{\lambda}(A) = \left(C_*(A)\left[\left[u, u^{-1}\right]\right. / \left(uC_*(A)[[u]]\right), b + uB\right).$$

These are, respectively, the *negative cyclic*, the *periodic cyclic*, and the *cyclic* complexes of A over k.

Just to illustrate the usage of u: Cyclic periodic (bi)complex



1.1. **Chern character.** There exists in general a chern character(s)

- $ch : K_i^{alg}(A) \to CC_i^{-}(A)$
- $ch : K_i^{top}(A) \to CC_i^{per}(A)$

A formula for the second one is, say for an idempotent $e \in A$,

$$ch(e) = (e - \frac{1}{2}) \sum_n \frac{(2n)!}{n!} e^{\otimes 2n} \in CC_{even}^{per}(A)$$

1.2. **Pairings between chains and cochains.**

- Contraction

  $$\iota_D(a_0 \otimes \ldots \otimes a_n) = a_0 D(a_1, \ldots, a_k) \otimes a_{k+1} \otimes \ldots \otimes a_n$$

  In particular, $\iota_D \circ \iota_E = \iota_{E \cup D}$ and $[\iota_D, b] = \iota_{\delta D}$.
- Lie derivative

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum \pm a_0 \otimes \ldots \otimes D(a_{i+1}, \ldots) \otimes \ldots + \sum \pm D(a_{n-j+1}, \ldots, a_0, \ldots, a_i) \otimes a_{i+1} \otimes \ldots$$

  In particular, $b = \pm L_m$.

2. Calculi

Definition 5. *A calculus is a pair of graded vector spaces* $(\mathcal{A}^*, \Omega^*$ *such that*

- $\mathcal{A}^*$ *is a Gerstenhaber algebra*
- Ω^{-*} *is a graded module over the graded commutative algebra* $\mathcal{A}^*$ *under an action* $(a, \omega \to \iota_a \omega$
- Ω^{-*} *is a graded module over the graded Lie algebra* $\mathcal{A}^{*+1}$ *under an action* $(a, \omega \to L_a \omega$
- *There is given an operator* $d : \Omega^* \to \Omega^{*+1}$ *satisfying*
  - $d^2 = 0$
  - $[L_a, \iota_b] = \iota_{[a,b]}$
  - $L_[a, b] = L_a \iota_b + (-1)^{|a|} \iota_a L_b$
  - $[d, \iota_a] = L_a$

Example 6. (1) *For a commutative algebra A,* $(\Lambda^*_A Der(A), \Omega)^*_{A/k})$ *is a calculus.*
(2) *For a smooth manifold X,* $(\Gamma(X, \Lambda^* T_X), \Omega^*(X)$ *is a calculus.*
(3) *The operations* ι_D*,* L_D*, B, cup product and the Gerstenhaber bracket turn the pair* $H^*(A, A), H_*(A, A))$ *into a calculus.*
(4) *In the case of A regular commutative or* $A = C^\infty(X)$*,* I_{HKR} *induces an isomorphism of calculi.*

2.1. **Formality for chains.** The following holds on the level of chains.

Theorem 7. *For any associative algebra, there is a naturally defined DG calculus* $(\mathcal{C}^*(A), \mathcal{C}_*(A))$ *together with quasiisomorphisms*

- $\mathcal{C}^*(A) \to C^*(A, A)$ *of DGLA*
- $(\mathcal{C}_*(A), d) \to (C_*(A), B)$ *of DGL-modules.*

Theorem 8 (Formality theorem for chains)**.** *For a regular commutative algebra A there exist a quasiisomorphism of DG calculi*

$$(\mathcal{C}^*(A), \mathcal{C}_*(A)) \to (\Lambda^*_A Der(A), \Omega)^*_{A/k})$$

In the case when $A = C^\infty(X)$*, there exists a quasiisomorphism of calculi*

$$(\mathcal{C}^*(A), \mathcal{C}_*(A)) \to (\Gamma(X, \Lambda^* T_X), \Omega^*(X)$$

Fact 9. (1) *The DG calculus* $(\mathcal{C}^*(A), \mathcal{C}_*(A))$ *is constructed naturally in A, but the construction depends on a choice of a Drinfeld associator and is inexplicit*
(2) *There is a visible part of "NC"-calculus, which are given by explicit formulas - for example the Cartan formulas*

(3) *For any Gerstenhaber algebra $\mathcal{A}^*$ one can construct its enveloping algebra $\mathcal{Y}(\mathcal{A}^*)$ with generators ι_a, L_a, $|\iota_a| = |a|$, $|L_a| = |a| - 1|$ and relations $\iota_a \iota_b = \iota_{ab}$, $[L_a, L_b] = L_[a,b]$, $[L_a, \iota_b] = \iota_[a,b]$ and $L_{ab} = L_a \iota_b + (-1)^{|a|}\iota_a L_b$. It has a differential $\iota_a \to L_a \to 0$. The algebra $\mathcal{Y}(\mathcal{A}^*)$ is, up to homotopy, independent of the choice of an associator.For example, $\mathcal{Y}(\Gamma(X, \Lambda^* T_X))$ coincides with the algebra of differential operators on X.*

2.2. **Applications of formality for chains.** We will work out the case of deformations of smooth manifold X. The formality theorem for chains provides the following picture

$$C^{*+1}(C^\infty(X), C^\infty(X)) \longleftarrow \mathcal{C}^{*+1}(C^\infty(X)) \longrightarrow \Lambda^{*+1}T_X$$

$$D \longleftarrow\!\shortmid \Pi \mapsto \pi$$

$$\delta D + \tfrac{1}{2}[D, D] = 0 \qquad \delta\Pi + \tfrac{1}{2}[\Pi, \Pi] = 0 \qquad [\pi, \pi] = 0]$$

and a compatible quis's on the level of modules

$$C_{-*}(C^\infty(X))[[u]] \longleftarrow \mathcal{C}_{-*}(C^\infty(X))[[u]] \longrightarrow \Omega^{-*}(X)[[u]]$$

$$b + uB \longleftarrow\!\shortmid b + ud \mapsto ud$$

Here D is a Maurer-Cartan element in $tC^2(C^\infty(X), C^\infty(X))[[t]]$. Π and π are essentially uniquely determined by Goldman-Nilsson theorem. D gives a deformation of $C^\infty(X)[[t]]$ with the product

$$a * b = ab + D(a, b)$$

We will denote it by A_π. Now note that $C_*(A_\pi) = C_*(C^\infty(X))[[\hbar]], b + L_D$, hence we get quis's

$$(C_{-*}(A_\pi)[[u]], b + uB) \longleftarrow \bullet \longrightarrow (\Omega^{-*}[[t, u]], L_\pi + ud)$$

If we invert u, the isomorphism $\exp(\frac{\iota_\pi}{u})$ intertwines $L_\pi + ud$ and ud. Hence we get quis's

$$(C_{-*}(A_\pi)((u)), b + uB) \longleftarrow \bullet \longrightarrow (\Omega^{-*}[[t, u]]_u, ud)$$

Definition 10. *The resulting quis $TR : (C_{-*}(A_\pi)((u)), b + uB) \longrightarrow (\Omega^{-*}[[t, u]]_u, ud)$ is called the canonical trace.*

3. Index theorems

In the language of previous section, the following holds.

Theorem 11 (Index theorem)**.** *The following diagram is commutative*

$$\begin{array}{ccc} (C_{-*}(A_\pi)[[u]], b_\pi + uB) & \xrightarrow{TR} & (\Omega^{-*}((t,u)), ud) \\ \Big\downarrow{\scriptstyle t=0} & & \Big\uparrow{\scriptstyle \cup\sqrt{Td(T_X)}} \\ (C_{-*}(C^\infty(X))[[u]], b + uB) & \xrightarrow[I_{HKR}]{} & (\Omega^{-*}((t,u)), ud) \end{array}$$

3.1. **Atiyah Singer index theorem.** The simplest case of an application of above is as follows. Let X be a smooth manifold and set $M = T^*X$.

Deformation

We will denote by $S(M)$ the space of Schwartz functions. Define

$$\mathrm{Op} : S(M) \to \mathrm{End}\,(C^\infty(X))$$

by

$$\mathrm{Op}(\theta)\,(u)\,(x) \;=\; \int_{T_z^*X} d\xi \int dy\,\chi(x,y)\,\mathrm{e}^{i\,\xi.v}\,\theta(z,\xi)\,u(y)$$

where $\chi(x,y)$ is a cutoff function, equal to one in a neighbourhood of the diagonal and such that the inverse of the exponential map $T^*X \times X \to X$ is well defined in its support, z is the midpoint of the geodesic joining x and y and $exp_z\frac{v}{2} = y$. Conversely, if P is in $\mathrm{End}\,(C^\infty(X))$, set

$$\sigma(P)(z,\xi) \;=\; P_y(\,\chi(x,y)\exp^{i\,\xi.v}\,)|_{x=y=z}$$

Set, for $\theta \in C^\infty(T^*X)$, $\theta_t(x,\xi) \;=\; \theta(x, t\xi)$. Then

$$t \to \sigma(\,\mathrm{Op}(\theta_t^1) \circ \mathrm{Op}(\theta_t^2)\,)_{t^{-1}}(x,\xi)$$

has an asymptotic expansion at $t = 0$, which defines a $*$-product on M, i. e. a deformation $(C^\infty(M)[[t]], *)$ and such that $[f,g] = t\{f,g\} + o(t)$ Let us denote the corresponding algebra by $\mathcal{O}_M^t$.

Trace

The usual trace on the trace class operators on $L^2(M)$ has an asymptotic expansion which gives a (unique) trace τ on $\mathcal{O}_M^t[t^{-1}]$ such that

$$Tr(\,\mathrm{Op}(\theta_t)) \sim_{t\to 0} \tau(\theta) = \frac{1}{t^n}(\int_M f\omega^n + o(t)).$$

We will think of τ as an element of $CC^{ev}_{per}(\mathcal{O}^t_M)$ and, as such, it coincides with the composition

$$CC^{ev}_{per}(\mathcal{O}^t_M) \xrightarrow{\quad TR \quad} \Omega^*_M \xrightarrow{\quad \int_M \quad} \mathbb{C}.$$

Remark 12. *The singularity in the trace at $t = 0$ is responsible for Bott periodicity.*

Given a "symbol" $\sigma \in C^\infty(M)$ (symbol means that it has a nice asymptotic expansion at infinity), it is elliptic if it is invertible modulo $C_0(M)$. Then $\mathrm{Op}(\sigma_t)$ is Fredholm (invertible modulo compact operators) and

$$Index\ \mathrm{Op}(\sigma_t) = dim\mathrm{Ker}\ \mathrm{Op}(\sigma_t) - dim\mathrm{Coker}\ \mathrm{Op}(\sigma_t) \in \mathbb{Z}$$

makes sense and is constant in t. Then

$$Index\ \mathrm{Op}(\sigma) =< \tau, ch(\sigma) >$$

where the chern character on the right hand side is computed in $CC^{per}_{ev}(\mathcal{O}^t_M)$. The algebraic index theorem applied to this formula gives

Theorem 13 (Atiyah Singer)**.**

$$Index\ \mathrm{Op}(\sigma) = \int_M ch_0(\sigma)\hat{A}_M$$

where ch_0 is the usual chern character with values in (Ω^*_M, d)).