LECTURE 1

RYSZARD NEST

Contents

1. Hochschild cochains

Notation 1.

- *k-a commutative algebra with unit, $\mathbb{Q} \subset k$*
- *A-a flat k-algebra with unit*
- *$C^n(A,A) = Hom_k(A^{\otimes n}, A)$ - Hochschild cochains*
- *$\delta : C^n(A,A) \to C^{n+1}(A,A)$,*

$$
\begin{aligned}
(\delta D)(a_1 \dots, a_{n+1}) = a_1 D(a_2 \dots, a_{n+1}) + \\
\sum_{i=1}^{n} (-1)^i D(a_1 \dots, a_i a_{i+1}, \dots a_{n+1}) \\
+(-1)^{n+1} D(a_1 \dots, a_n) a_{n+1}
\end{aligned}
$$

Lemma 2. *$\delta^2 = 0$. The groups $H^n(A,A) = \frac{ker(\delta : C^n \to C^{n+1})}{im(\delta : C^{n-1} \to C^n)}$ are called the Hochschild cohomology groups. In particular*

- *$H^0(A,A) = Z(A)$, the center of A*
- *$H^1(A,A) = Der(A)/ad(A)$*
- *$H^2(A,A) = \dfrac{\text{Infinitesimal deformations of } A}{\text{isomorphisms}}$,*

Here, an infinitesimal deformation of A is an associative $k[\epsilon]$-linear product $*$ on $A[\epsilon]/\epsilon^2$ which coincides with the original product on A modulo ϵ and an isomorphism is a $k[\epsilon]$-linear map of the form $T(a) = a + X(a)\epsilon$ which intertwines the $*$-products.

1.1. **Algebraic structures on Hochschild cochains I.** Given $D \in C^n$ and $E \in C^m$, the cup product $D \cup E \in C^{n+m}$ is given by

$$D \cup E(a_1 \dots a_{n+m}) = D(a_1 \dots a_n)E(a_{n+1} \dots a_{n+m})$$

Lemma 3. *$(C^*(A,A), \cup, \delta)$ is a differential graded algebra. The cup product is commutative up to homotopy, in fact the following holds*

$$D \cup E - (-1)^{|D||E|} E \cup D = -(-1)^{|D||E|}\delta(D\{E\}) - (\delta D)\{E\} - (-1)^{|D|-1} D\{\delta E\}.$$

Here the brace is defined by

$$D\{E\}(a_1, \dots, a_{n+m-1}) = \sum_{i=0}^{n} (-1)^{i(m-i)} D(a_1, \dots, a_i, E(a_{i+1}, \dots, a_{i+m}), \dots, a_{n+m-1})).$$

Hence $H^*(A,A)$ is a graded commutative algebra.

Lemma 4. *The bracket*

$$[D,E] = D\{E\} - (-1)^{(|D|-1)(|E|-1)} E\{D\}$$

defines on $C^{*+1}(A,A)$ a structure of DGLA (differential graded Lie algebra).

Proof. The brace $D\{E\}$ is not quite associative, but

$$(D\{E\})\{F\} - D\{E\{F\}\} = D\{F,E\} - (-1)^{(|E|-1)(|F|-1)} D\{E,F\},$$

where

$$D\{E,F\}(a_1, \dots) = \sum_{i<j} (-1)^{(|E|-1)i + (|F|-1)j} D(a_1 \dots, E(a_i, \dots), \dots, F(a_j, \dots), \dots).$$

Hence

$$\begin{aligned} &[D,[E,F]] + \circlearrowleft \quad = \\ &(D\{E\})\{F\} - D\{E\{F\}\} - D\{F,E\} \pm D\{E,F\} + \circlearrowleft \end{aligned}$$

cancel out.

Let $*$ be any product on A and set $M(a,b) = a * b$. Then

$$\text{M is associative} \iff [M,M] = 0.$$

In particular, if $m(a,b) = ab$, then $\delta D = (-1)^{|D|}[m,D]$ and $\delta^2 = 0$ follows from $[m,m] = 0$ and the Jacobi identity

$$\delta[D,E] = [\delta D, E] + (-1)^{|D|-1}[D, \delta E].$$

Hence

- $C^{*+1}(A,A)$ is a DGLA
- $H^{*+1}(A,A)$ is a Lie algebra

Moreover,

$$[D, E \cup F] = [D, E] \cup F + (-1)^{(|D|-1)|E|} E \cup [D, F] + \delta D\{E, F\}$$

Recall

Definition 5. *A Gerstenhaber algebra is a graded vector space A^* with binary operations m and $[\cdot,\cdot]$, such that*

- (A^*, m) *is a graded commutative algebra*
- $(A^{*+1}, [\cdot,\cdot])$ *is a Lie algebra*
- $[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c]$

The above discussion says that $H^*(A, A)$ has the structure of a Gerstenhaber algebra.

example 6. *Let A be a commutative algebra. Then $Der A$ is an A-bimodule and*

$$\Lambda^*_A Der(A)$$

is a graded commutative algebra (with the wedge product). We set

- $[v, a] = v(a)$ *for* $a \in \Lambda^0_A(A) = A$ *and* $v \in Der(A) = \Lambda^1_A Der(A)$
- $[v, w] = vw - wv$ *for* $v, w \in Der(A)$

$[\cdot,\cdot]$ *has a unique extension to* $\Lambda^*_A Der(A)$ *such that* $(\Lambda^*_A Der(A), \wedge, [\cdot,\cdot])$ *is a Gerstenhaber algebra. The map*

$$\Lambda^n_A Der(A) \ni v_1 \dots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} v_{\sigma_1} \cup \dots v_{\sigma_n} \in C^n(A, A)$$

is a morphism of complexes $I_{HKR} : ((\Lambda^*_A Der(A), 0) \to (C^*(A, A), \delta)$.

example 7. *Similarly, for $C^\infty(X)$ (X a smooth manifold) $\Gamma(X, \Lambda^* T_X)$ is a Gerstenhaber algebra. The bracket here is called the Schouten-Nijenhuis bracket. One can obviously replace polyvector fields on a smooth manifold by the sheaf of holomorphic polyvector fields on an analytic manifold.*

In this example, if set $A = C^\infty(X)$ and from now on $C^*(A, A)$ will be the cochains given by polydifferential operators, i. e. of the form

$$D(f_1, \dots, f_n) = \sum^{finite} D_1(f_1) \dots D_n(f_n),$$

where D_i are differential operators. We again get a morphism of complexes

$$I_{HKR} : ((\Lambda^*_A Der(A), 0) \to (C^*(A, A), \delta).$$

Theorem 8 (Hochschild, Kostant, Rosenberg)**.** *For A regular commutative algebra, or $A = \mathbb{C}^\infty(X)$ (or sheaf of holomorphic functions...) I_{HKR} is a quasiisomorphism.*

Note that I_{HKR} preserves neither the product, nor the bracket. But the induced maps

$$(\Lambda^*_A Der(A), \wedge, [\cdot,\cdot]) \to (H^*(A,A), \cup, [\cdot,\cdot])$$

is an isomorphism of Gerstenhaber algebras.

1.2. **Formality.**

Theorem 9 (Tamarkin)**.** *For an associative algebra A there exists a DG Gerstenhaber algebra $\mathcal{C}^*(A)$, natural in A and*

1. *a DGLA quasiisomorphism $\mathcal{C}^*(A) \to C^*(A,A)$*
2. *for A regular commutative or $C^\infty(X)$ (or...) there exists a quasiisomorphism of DG Gerstenhaber algebras*

$$\mathcal{C}^*(A) \to \Lambda^*_A Der(A).$$

Remark 10. *In other words, Hochschield cochains carry a structure of a DG Gerstenhaber algebra, if one chooses the model "correctly". This should be compared to the algebra of singular cochains on a topological space which carries a cup-product commutative up to cohomology. But if one chooses instead the model (over $\mathbb{Q}$) of Sullivan forms, it becomes graded commutative on the level of cochains.*

A DGLA $\mathcal{A}^*$ is formal, if there exists a chain of DGLA quasiisom. of the form

$$\begin{array}{ccc} & \mathcal{B}^* & \\ \swarrow & & \searrow \\ \mathcal{A}^*_i & & \mathcal{A}^*_{i+1} \end{array}$$

with $\mathcal{A}^*_i = \mathcal{A}^*$ and $\mathcal{A}^*_n = H^*(\mathcal{A}^*)$. As a "corollary"

Theorem 11 (Kontsevich)**.** *$C^\infty(X)$ is formal.*

1.3. **Braces.**

For $D, E_1, \ldots, E_n \in C^*(A,A)$, we can form braces

$$D\{E_1,\ldots,E_n\}(a_1,\ldots,a_N) = \sum \pm D(a_1,\ldots,E_1(a_{i_1+1},\ldots,a_{i_{i_1+n_1}}), a_{\_\_\_}, E_2(\_\_\_),\ldots, \ldots, E_n(\_\_\_),\ldots,a_N)$$

The sum is over all possible insertions of E_k's, where the order of a's and E's is preserved. The sign is dictated by the rule:

transposition $a \leftrightarrow E$ contributes the sign $(-1)^{(|a|-1)(|E|-1)}$

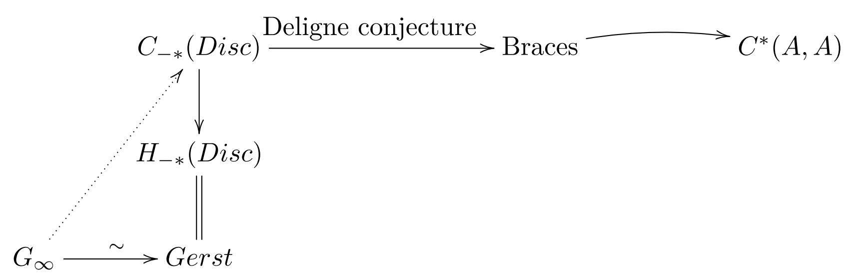
Lemma 12 (Brace relations)**.**

$$(D\{E_1,\dots,E_m\})\{F_1,\dots F_n\} = \sum \pm D\{F_1,\dots,E_1\{F_{i_1+1},\_\_\},\dots,E_2\{F,\_\_\}\dots, \dots,E_m\{F_{i_m+1},\_\_\},\dots,F_n\}$$

Recall what we used already:

- $\delta = [m,\cdot]$ respects $\cup$ and $[\cdot,\cdot]$, where $D\cup E = (-1)^{|D|}m\{D,E\}$
- $[\cdot,\cdot]$ is a graded Lie bracket;
- $[D, E\cup F] \sim [D,E]\cup F \pm E\cup[D,F]$ with the homotopy given by $D\{E,F\}$.

The basic result says that $C^*(A,A)$ can be given a structure of a DG Gerstenhaber algebra, up to a quis of DGLA's. The idea of the proof is as follows.



G_∞ is a cofibrant resolution of the operad $Gerst$, and the dotted map exists by general principles. As a corollary, $C^*(A,A)$ is a G_∞-algebra and DG Gerstenhaber algebra $\mathcal{C}^*(A)$ is its rectification.

From algebra to topology

$$\text{Brace operations act on } C^*(A,A)$$
$$\Downarrow$$
$$\text{Action } C_{-*}(Disc(n))\otimes C^*(A,A)^{\otimes n} \to C^*(A,A)$$

Existence of the bottom action is the "Deligne conjecture".

Back from topology to algebra

- $H_{-*}(Disc(n)$ is the natural operad $Gerst(n)$ of n-ary operations on a Gerstenhaber algebra (Arnold, Cohen)
- $C_{-*}(Disc(n))) \to H_{-*}(Disc(n)) = Gerst(n)$ is a quasiisomorphism of operads, i.e. the chain operad of the little disc operad is formal (Tamarkin) - this step involves an associator.

1.4. **Applications to deformation theory.** A deformation of an associative algebra A is an associative, $k[[t]]$-linear product $*$ on $A[[t]]$ of the form

$$a * b = ab + \sum_{i>0} t^i D_i(a,b),\ \ D_i \in C^2(A,A).$$

An isomorphism of two deformations $(A[[t]], *)$ and $(A[[t]], *')$ is a $k[[t]]$-linear bijection $T : A[[t]] \to A[[t]]$ of the form

$$T(a) = a + \sum_{i>0} t^i T_i(a,b),\ \ T_i \in C^1(A,A)$$

satisfying $T(a * b) = T(a) *' T(b)$. If we set $a * b = ab + D(a,b)$, then D satisfies $[m + D, m + D] = 0$, or

$$\delta D + \frac{1}{2}[D,D] = 0.$$

So

Definition 13. *Let $(\mathfrak{g}^*, d)$ be a pronilpotent DGLA. A Maurer-Cartan element of $\mathfrak{g}$ is an element $\omega \in \mathfrak{g}^1$ satisfying the Maurer-Cartan equation*

$$d\omega + \frac{1}{2}[\omega,\omega] = 0.$$

two Maurer-Cartan elements ω_1 and ω_2 are gauge equivalent, if there exist an element $X \in \mathfrak{g}^0$ satisfying

$$d + \omega_2 = e^X (d + \omega_1) e^{-X}.$$

We set

$$Def(\mathfrak{g}) = \frac{\textit{Maurer Cartan elements}}{\textit{gauge equivalence}} (= \pi_0(\mathfrak{g}))$$

Theorem 14 (Goldmann-Nilsson, Yekuteli)**.** *A quasiisomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ of pronilpotent DGLA's induces a bijection $Def(\mathfrak{g}) \to Def(\mathfrak{h})$.*

Corollary 15. *For a regular, commutative algebra A (or $A = C^\infty(X)$ or...)*

$$\begin{array}{ccc} Def(C^{*-1}(A,A)) & \xleftarrow{\ \simeq\ } & Def(\mathcal{C}*(A)) \\ & & \big\downarrow \simeq \\ & & Def(\Lambda^*_A Der(A))). \end{array}$$

In particular, we get a bijection

$$\frac{\textit{deformations of } C^\infty(X)}{\textit{isomorphisms}} \simeq \frac{\textit{formal Poisson structures } t\pi_1 + t^2\pi_2 + \dots}{\textit{formal diffeomorphisms}}$$

We get more precise information. Given $\pi = \sum_1^\infty t^n \pi_n$, $[\pi,\pi]_{Sch} = 0$, let $(A_\pi, *)$ be the corresponding algebra. Then we get

$$C^{*+1}(A,A) \longleftarrow \mathcal{C}^{*+1}(A) \longrightarrow \Lambda_A^{*+1} Der(A)$$

$$D \longleftarrow \Pi \longrightarrow \pi$$

where $D \in tC^2(A,A)$ satisfies

$$\delta D + \frac{1}{2}[D,D] = 0,\ a * b = ab + D(a,b)$$

In particular, we get the following

Corollary 16. *There exists a chain of quasiisomorphisms of DGLA's*

$$C^{*+1}(A_\pi, A_\pi) \longleftarrow \mathcal{C}^{*+1}(A)[[t]] \longrightarrow \Lambda_A^{*+1} Der(A)[[t]]$$

$$[m_\pi, \cdot] \longleftarrow \delta + [\Pi, \cdot] \longrightarrow [\pi, \cdot]$$

In particular,

$$Z(A_\pi) \simeq \{a \in A[[t]] \mid \pi(da, \cdot) = 0\}$$

Remark 17. *[Duflo isomorphism] The above isomorphism of the center of A_π with the Poisson center of π is as vector spaces.*

2. Deformation 2-groupoids

Let $(\mathfrak{g}, [,], d)$ be a nilpotent DGLA starting from dimension -1:

$$\mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \dots .$$

As above, a Maurer-Cartan element $\mu \in \mathfrak{g}^1$ satisfies the equation

$$d\mu + \frac{1}{2}[\mu,\mu] = 0.$$

We will think of it as a flat connection

$$\nabla_\mu = d + [\mu,]$$

and will denote the set of Maurer-Cartan elements by $MC(\mathfrak{g})$ $\mathfrak{g}^0$ is a nilpotent Lie algebra and we will denote the corresponding Lie group by G^0 - think of it as the gauge group. G^0 acts on the space of flat connections by "gauge transformations":

$$d + ad\mu \to Ad(e^X)(d + ad\mu).$$

This descends to an action on Maurer Cartan elements, by

$$Ad(e^X)(d + ad\mu) = d + ad(\int_0^1 e^{adtX}(dX)dt + e^{adX}(\mu)).$$

Given μ, the bracket

$$[\theta, \phi]_\mu = [\nabla_\mu \theta, \tau]$$

defines a structure of Lie algebra on $\mathfrak{g}^{-1}$, and the corresponding group G^{-1}_μ acts on G^0 by multiplication by $e^{\nabla_\mu \theta}$.

All together, we get a Deligne two-groupoid $MC^2(\mathfrak{g})$:

- Objects - Maurer Cartan elements μ.
- 1-morphisms e^X, $X \in \mathfrak{g}^0$, acting by $\mu \to \int_0^1 e^{adtX}(dX)dt + e^{adX}(\mu)$.
- 2-morphisms acting on $Hom(\mu_1, \mu_2)$ by multiplication by $e^{\nabla_{\mu_2}\theta}$ for $e^\theta \in G^{-1}_{\mu_2}$.

Theorem 18. *A L_∞ quasiisomorphism of two DGLA's vanishing in degrees below -1 induces an equivalence of the associated Deligne two groupoids.*

In particular, the formality of, say, $C^*(C^\infty(X), C^\infty(X))$ says that the Deligne two groupoid of deformations of $C^\infty(X)$ is equivalent to the one, where

- objects are formal Poisson structures $\pi \in t\Lambda^2 T_X$
- 1-morphisms are the formal diffeomorphisms $exp(X)$, $X \in t\Lambda^1 T_X$
- 2-morphisms are the formal diffeomorphisms $exp(X_\theta)$ associated to Hamiltonian vector fields $[\pi, \theta]$, $\theta \in C^\infty(X)[[t]]$

To be more precise, let us define equivalence of two-groupoids of the form MC^2. Given a DGLA $\mathfrak{g}$ as above,

$$\Sigma(\mathfrak{g}) = \{n \to MC(\mathfrak{g} \otimes \Omega^*(\Delta_n)\}$$

is a Kan simplicial set with homotopy vanishing in dimensions above 2, and two MC^2's are equivalent if the corresponding Σ's are homotopy equivalent.