Ma Solleveld : p-adics I

What are p-adic groups? Where do they show up?

$GL_n(\mathbb{Q})$ reps related to $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ reps

Some kinds of reps: - algebraic: on $\mathbb{Q}^n$ by matrix multiplication
- continuous: restrictions of $GL_n(\mathbb{R})$-reps, e.g. on $L^2(\mathbb{R}^n)$ via $f \mapsto f \circ A$
- number theoretic: e.g. $\pi: A \mapsto |\det A|_p \in \mathbb{R}_{>0}$
  where $|\cdot|_p$ is the p-adic absolute value $|p^n \frac{a}{b}|_p := p^{-n}$, a,b coprime to p
  $|0|_p := 0$

Claim: π does not extend continuously to $GL_n(\mathbb{R})$: $0 = \lim_{n\to\infty} p^{n}$ but $\infty = \lim_{n\to\infty} |p^{n}|_p$

To find all (irred.) reps of $GL_n(\mathbb{Q})$, one has to consider all embeddings

$$\begin{array}{ccccc} & & GL_n(\mathbb{Q}) & & \\ \swarrow & \downarrow & & \searrow & \\ GL_n(\mathbb{Q}_2) & GL_n(\mathbb{Q}_3) & \cdots & & GL_n(\mathbb{R}) \end{array}$$

Similarities + Differences between Lie-/p-adic groups: Lie groups are mfds.

What's $GL_n(\mathbb{Q}_p)$ as top. space? On $GL_n(\mathbb{Q}_p)$: metric $d(A,B) = \max_{i,j} |a_{ij} - b_{ij}|_p$

Lemma: $(GL_n(\mathbb{Q}_p), d)$ is totally disconnected (connected components are points).

Proof: Let $Y \subset GL_n(\mathbb{Q}_p)$ connected, $\neq \emptyset$, $A \in Y$, $d_A : Y \to \mathbb{R}$, $B \mapsto d(A,B)$ continuous.

$d_A(Y)$ connected because Y is connected. But $\operatorname{im} d_A \subset \{0\} \cup \{p^n\}_{n\in\mathbb{Z}}$

$\Rightarrow d_A(Y) = 0 \Rightarrow Y = \{A\}$. ☐

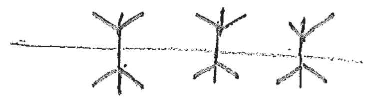
Properties of $GL_n(\mathbb{Q}_p)$:
1.) complete metric space (w/ correct metric)
2.) totally disconnected, not discrete
3.) locally compact
4.) alg. group (multi + inverse are algebraic fcts of the entries)
5.) top group w.r.t. $|\cdot|_p$.

Examples: $SO_n(\mathbb{Q}_p)$, $SL_n(\mathbb{Q}_p)$, $Sp_{2n}(\mathbb{Q}_p)$

Comparison w/ Lie groups: 1.) (w/ correct metric), 3.), sometimes 4.), 5.), never 2.) unless finite.

Lie groups arise often as symmetries of something: $O_n(\mathbb{R}) = \{$isometries of $S^{n-1}\}$
$PGL_2(\mathbb{R}) = \{$isometries of $\mathbb{H}\}$

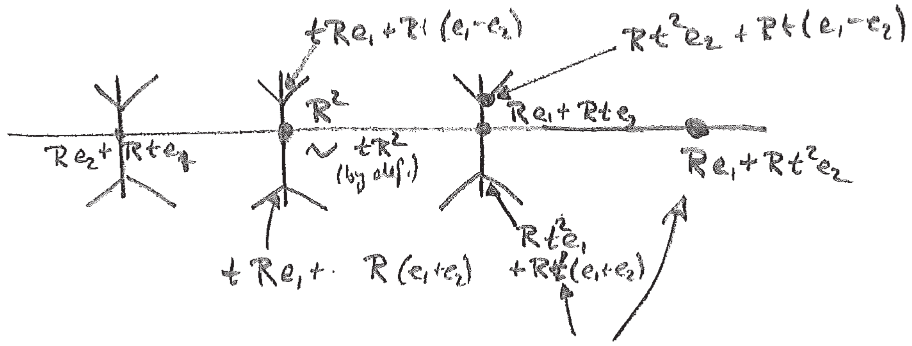
p-adic groups arise as isometries of (p+1-regular) trees



$p=3$

$p+1 = \#$ lines in $\mathbb{F}_p^2$, $R = \mathbb{F}_p[[t]]$. $p+1 = \#$ R-submodules L of R^2 of index p.

submod. $L \supset tR^2 \to R^2/tR^2 = \mathbb{F}_p^2$



neighborhoods of L : R-submodules of L of index p modulo $L \sim t^n L$ $\forall n \in \mathbb{N}$.

$\Rightarrow$ there exists a bijection {vertices of tree} $\longleftrightarrow$ {R-submodules of ?
of index p^n

/.

$\Rightarrow$ $Gl_2(R)$ acts on tree

Replace R by quotient field $F = \mathbb{F}_p[[t]][t^{-1}] = \mathbb{F}_p((t))$

Replace submodules of R^2 by R-lattices in F^2.

Lattice in F^n : compacts + discrete quotients, R-submodule

(same as $Rv + Rw$, $v, w \in F^2$ lin. indep.)

Homothety of lattices $L \sim L' \Leftrightarrow L = aL'$ for some $a \in F^*$

$\Rightarrow$ bijection {vertices of tree} $\longleftrightarrow$ {R-lattices in F^2}$/_\sim$

$\rightsquigarrow$ $Gl_2(F)$ acts on T

Stabilizer of $[R^2]$ is $GL_2(R) \cdot Z(GL_2(F))$

M. Solleveld: p-adics II

Valuated fields: **Def:** An absolute value on a field F is a map $|\cdot|: F \to \mathbb{R}_{\geq 0}$ s.t.

- $|xy| = |x||y|$
- $|x+y| \leq |x|+|y|$ $\quad \forall x,y \in F$.
- $|x| = 0 \iff x = 0$

$|\cdot|$ nonarchimedean if $|x+y| \leq \max\{|x|,|y|\}$.
$|\cdot|$ archimedean if $\forall x,y \in F, x \neq 0\ \exists n \in \mathbb{N} : |nx| > |y|$.

Examples: $|\cdot|, |\cdot|^{1/2}$ on $\mathbb{R}, \mathbb{C}$, $|\cdot|_p$ on $\mathbb{Q}_p$

$\rightsquigarrow$ metric on F, $d(x,y) = |x-y|$

Def: A discrete valuation on F is a surj. map $v: F^* \to \mathbb{Z}$ s.t. $v(xy) = v(x)+v(y)$
$v(0) := +\infty$.
and $v(x+y) \geq \min\{v(x), v(y)\}$ $\forall x,y \in F^*$

Examples:
- $F = \mathbb{Q}_p$, $v(p^n x) = n$ if $x \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$.
- $F = E[[t]][t^{-1}] = E((t))$, E field, $v(a_n t^n + a_{n+1}t^{n+1} + \dots) = n$ if $a_n \neq 0$.
- $\mathcal{O} = \{x \in F : v(x) \geq 0\}$ subring of F.

$\mathcal{O}^* = \{x \in F : v(x) = 0\}$ invertibles in $\mathcal{O}$

$m = \{x \in F : v(x) > 0\}$ unique maximal ideal in $\mathcal{O}$

$\mathcal{O}/m$ residue field of F.

for $\mathbb{Q}_p$: $\mathcal{O} = \mathbb{Z}_p$, $m = p\mathbb{Z}_p$, $\mathcal{O}/m = \mathbb{F}_p$.

for $E((t))$: $\mathcal{O} = E[[t]]$, $m = tE[[t]]$, $\mathcal{O}/m = E$.

$|x|_v := e^{-v(x)}$ is a nonarchimedean absolute value on F (can choose other number >1 instead of e)
$\rightsquigarrow$ topology on F.

An element $\pi \in F$ w/ $v(\pi) = 1$ is called a uniformizer of F.

$\pi^n \mathcal{O} = \{x \in F : v(x) \geq n\}$ ball around $0 \in F$ of radius e^{1-n}
$\overset{!}{=} B(0, e^{1-n})$

$\{\pi^n \mathcal{O} : n \in \mathbb{N}\}$ basis of neighborhoods of $0 \in F$.

Prop: $\pi^n \mathcal{O}$ is compact $\iff$ $\mathcal{O}$ complete wrt. $|\cdot|_v$ and $\mathcal{O}/m$ finite

Proof: A metric space X is compact $\iff$ X is complete and totally bounded.
$\forall \varepsilon > 0\ \exists x_1, \dots, x_n \in X$: $X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$

"$\Rightarrow$": Suppose $\pi^n\mathcal{O}$ compact $\Rightarrow$ $\mathcal{O}$ compact ($\pi^n \cdot$ is homeomorphism)
$\Rightarrow$ Can cover $\mathcal{O}$ w/ finitely many balls of radius $1/2$, $B(x, \tfrac{1}{2}) = x+m$
$\Rightarrow$ $\mathcal{O} = \bigcup_{i=1}^{n} x_i + m \Rightarrow |\mathcal{O}/m| \leq n$.

"$\Leftarrow$": Suppose $\mathcal{O}$ complete, $\mathcal{O}/m$ finite.
$\Downarrow$
$\pi^n\mathcal{O}$ complete

Let $\varepsilon > 0$, For $e^{-d} < \varepsilon$, $B(x,\varepsilon) \supseteq x + \pi^{d}\mathcal{O}$, $[\pi^n\mathcal{O} : \pi^d\mathcal{O}] = \prod_{s=n+1}^{d} [\pi^{s-1}\mathcal{O} : \pi^s\mathcal{O}] \overset{!}{=} [\mathcal{O}:\pi\mathcal{O}]^{d-n} = |\mathcal{O}/m|^{d-n}$ $(n \leq d)$

$\Rightarrow \exists x_1, \dots, x_{|\mathcal{O}/m|^{d-n}}$: $\pi^n\mathcal{O} = \bigcup x_i + \pi^d\mathcal{O} \subseteq \bigcup B(x_i, \varepsilon)$ ∎

In particular: $\mathcal{O} = \mathbb{Z}_p$ is compact (because $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ wrt. $|\cdot|_p$)

$\mathcal{O} = E[[t]]$ compact $\iff$ E finite.

((Much of the theory continues to hold for infinite E, just can't use local compactness anymore))

Local fields: Def: A *local field* is a top. field which is *locally compact* (also Hausdorff) but not discrete.

Examples: $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_q((t))$

Def: (normalized absolute value) Let μ Haar measure on F. μ is unique up to multiplication w/ elements of $\mathbb{R}_{>0}$. (for addition)

$A \longmapsto \mu(xA)$, $x \in F^*$, is again a Haar measure $\Rightarrow \exists\, |x|_F : \mu(xA) = |x|_F \mu(A)$ $\forall$ measurable $A \subset F$.

$|\cdot|_F$ is an absolute value on F.

Example: $F = \mathbb{Q}_p$, $A = \mathbb{Z}_p$, $\mu(A) := 1$.

$$\frac{\mu(x\mathbb{Z}_p)}{\mu(\mathbb{Z}_p)} = \frac{\mu(p^{\nu(x)}\mathbb{Z}_p)}{\mu(\mathbb{Z}_p)} \quad \text{because } y := x p^{-\nu(x)} \in \mathbb{Z}_p^\times, \text{ but } y\mathbb{Z}_p = \mathbb{Z}_p$$

$$= [p^{\nu(x)}\mathbb{Z}_p : \mathbb{Z}_p] = p^{-\nu(x)} = |x|_p$$
$(\nu(x) \leq 0)$

$\Rightarrow |\cdot|_{\mathbb{Q}_p} = |\cdot|_p$.

Lemma: Let F be a local field, E/F finite extension $\Rightarrow$ E also local field.

Proof: $E \cong F^n$ as F-vector space, so E w/ the product topology from F^n is locally compact. $\square$

Every $x \in E$ induces a F-linear map $E \longrightarrow E$, $y \longmapsto xy$ w/ determinant $\det_{E/F}(x) \in F$

Exercise: $|x|_E = |\det_{E/F}(x)|_F$

Special cases: $z \in \mathbb{C}$: $|z|_{\mathbb{C}} = |z\bar{z}|_{\mathbb{R}} = |z|^2$, $|\cdot|^r \sim |\cdot|$ for $r>0$, defines same unit ball

Theorem: (Classification of local fields)

Let F be a local field. Then F is archimedean or non archimedean.

a) If F is archimedean $\rightarrow$ $F = \mathbb{R}, \mathbb{C}$.

b) If F is non archimedean $\Rightarrow$ $\exists$ discrete valuation,
If $\mathrm{char}(F) = 0$ $\Rightarrow$ $F =$ finite extension of $\mathbb{Q}_p$ for some prime p
If $\mathrm{char}(F) \neq 0$ $\Rightarrow$ $F \cong \mathbb{F}_q((t))$ w/ $\mathbb{F}_q$ finite field.

Def: A p-adic field is a nonarchimedean local field w/ residual characteristic p.

The multiplicative group of a p-adic field

$v : F^* \to \mathbb{Z}$ is continuous, so $\mathcal{O}^* = \{x \in F^* : v(x) = 0\}$ open + closed in F^*

$F^* \cong \mathcal{O}^* \times \mathbb{Z}$ (isomorphism of top. groups)

$a\pi^n \leftarrow (a, n)$, π uniformizer

Lemma: $\mathcal{O}^*$ is the unique maximal compact subgroup of F^*.

Proof: Suppose $K \subset F^*$ compact $\Rightarrow v(K) \subseteq \mathbb{Z}$ compact subgroup $\Rightarrow v(K) = \{0\}$

$\mathcal{O}$ is compact (above Prop + classification or use that $\{\pi^n \mathcal{O} : n \in \mathbb{N}\}$ is a neighborhood basis of $0 \in F$. F loc. comp $\Rightarrow \exists n : \pi^n \mathcal{O}$ compact $\Rightarrow \pi^n \mathcal{O}$ compact $\Rightarrow \mathcal{O}$ compact)

m is compact, $\mathcal{O}, m$ also open $\Rightarrow \mathcal{O}/m$ finite $\Rightarrow \mathcal{O}^* = \mathcal{O} \setminus m$ open and closed in $\mathcal{O}$

Since $\mathcal{O}$ is compact Hausdorff, so is $\mathcal{O}^* \subset \mathcal{O}$. ☐

Linear algebraic groups: Examples: GL_n, SL_n, SO_n, $Sp_{2n}(F) = \{A \in M_{2n}(F) : A^t J A = J\}$

$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$

$\mathcal{G}_m(F) := F^*$, $\mathcal{G}_a(F) = F$ multiplicative / additive groups.

Def: An algebraic group $\mathcal{G}$ (over $\bar{F}$) is
- an algebraic variety (over $\bar{F}$)
- a group
- multiplication + inverse are morphisms of algebraic varieties.

$\mathcal{G}$ is linear if it is a subgroup of GL_n for some n.

$I_{\mathcal{G}} := \{ f \in \bar{F}[x_{ij}, \det^{-1}] : f|_{\mathcal{G}} = 0 \}$ (x_{ij}: coord. fcts)

If $I_{\mathcal{G}}$ is generated by $F[x_{ij}, \det^{-1}] \cap I_{\mathcal{G}}$ $\Rightarrow$ $\mathcal{G}$ defined over F.

In this case, the group of F-rational points of $\mathcal{G}$ is $\mathcal{G}(F) = \{M \in GL_n(F) : f(M) = 0 \ \forall f \in F[x_{ij}, \det^{-1}] \cap I_{\mathcal{G}}\}$

Example: Unitary groups: E/F field extension of degree 2, char $(F) \neq 2$

$\Rightarrow$ Galois extn. w/ Galois group $\mathrm{Gal}(E/F) = \{id_E, \sigma\}$

$$\Rightarrow U_n(E/F) = \{M \in Gl_n(E) : \sigma(M)^t = M^{-1}\} = \mathcal{G}(F)$$

algebraic group defined over F

$U_n(E)$ will be different.

M. Solleveld: p-adic III

Main reference: Springer, Linear algebraic groups

Def: The Lie algebra of a linear algebraic group $G = \mathcal{G}(F)$ is the tangent space of G at I in $M_n(F)$

If $G = \{ M \in Gl_n(F) : f_i(M) = 0 \ \forall i \}$, then $\mathfrak{g} = \{ M \in M_n(F) : \langle (df_i)_I , M \rangle = 0 \ \forall i \}$

(more generally: $\mathfrak{g} = \mathrm{Der} \left(\underset{\uparrow}{\mathcal{O}}(G,1) / \mathcal{O}(G,1)^2 \right)$)

regular fcts vanishing at $1\!\!1$

The adjoint representation of G on $\mathfrak{g}$ is given by $Ad_A(X) = AXA^{-1}$.

Example: • $G = SL_n(F) = \{ A \in GL_n(F) : \det A = 1 \}$

$$\mathfrak{g} = \mathfrak{sl}_n(F) = \{ A \in M_n(F) : \langle d \det_{1\!\!1}, A \rangle = 0 \}$$

$$d \det_{1\!\!1} = \sum_{ij} \frac{\partial \det}{\partial x_{ij}} \Big|_{1\!\!1} dx_{ij} = \sum_{i=1}^n dx_{ii}$$

$$\Rightarrow \mathfrak{sl}_n(F) = \{ A \in M_n(F) : \langle \sum_{i=1}^n dx_{ii}, A \rangle \equiv \operatorname{tr} A = 0 \}$$

• Similar: $Lie(O_n(F)) = \mathfrak{o}_n(F) = \{ M \in M_n(F) : M^t = -M \}$

Exponential map: $\exp : \mathfrak{g} \longrightarrow G$

$$X \longmapsto \sum_{n=0}^\infty \frac{1}{n!} X^n \qquad \text{make sense of this!}$$

Need topology + division by $n!$ $\rightsquigarrow$ F local field of characteristic 0.

Lemma: Let F be a finite extension of $\mathbb{Q}_p$ and $G \subset GL_n(F)$ an algebraic group. For $X \in p^2 M_n(\mathcal{O}) \cap \mathfrak{g}$, $\exp(X)$ is well-defined.

Proof: $v_p(n!) = \prod_{m=1}^n v_p(m) \leq n$, $|n!|_p \geq p^{-n}$, $\left|\frac{1}{n!}\right|_p \leq p^n$

$$\Rightarrow \left| \frac{X^n}{n!} \right|_p \leq \left| \frac{p^{2n}}{n!} \right|_p \leq p^{-n} \Rightarrow \sum_{n=0}^\infty \frac{X^n}{n!} \text{ converges in } M_n(F) \ \blacksquare$$

On $p^2 M_n(\mathcal{O}) \cap \mathfrak{g}$, exp has the usual properties, in particular

$$\frac{d}{dt}\Big|_{t=0} \exp(tX) = X.$$

There is no exponential map over $\mathbb{F}_q((t))$.

$\rightsquigarrow$ Groups over $\mathbb{Q}_p$ are somewhat easier than over $\mathbb{F}_q((t))$.

Example: $\mathrm{Func}(SL_2(F)) = F[x_1,x_2,x_3,x_4] / (x_1x_4 - x_2x_3 - 1)$

$$\mathrm{Func}\Big(\underbrace{SL_2(F)/\{\pm \mathbb{1}\}}_{PSL_2(F)}\Big) = \mathrm{Func}(SL_2(F))^{\{\pm\mathbb{1}\}} = F[x_1,x_2,x_3,x_4]^{\text{even}} / (x_1x_4 - x_2x_3 - 1)$$

points of $PSL_2(F)$ = maximal ideals of $F[x_1,x_2,x_3,x_4]^{\text{even}}/(x_1x_4-x_2x_3-1)$:

- $\{A,-A\}$: $A \in SL_2(F)$
- e.g. for $F = \mathbb{Q}_p$ ($\sqrt{p}$ not a square in $\mathbb{Q}_p$!)

  still $B = \pm\begin{pmatrix} \sqrt{p} & 0 \\ 0 & \frac{1}{\sqrt{p}} \end{pmatrix} \in PSL_2(\mathbb{Q}_p)$ because $x_1^2(B) = p$, $x_1x_2(B) = 0$, $x_1x_4(B) = 1$

  $\Longrightarrow$ B defines an algebra homomorphism $\mathrm{Func}(PSL_2(F)) \to F = \mathbb{Q}_p$.

  $\Longrightarrow$ The quotient map $SL_2(\mathbb{Q}_p) \longrightarrow PSL_2(\mathbb{Q}_p)$ is not surjective. (Reason: $\mathbb{Q}_p$ not algebraically closed.)

Some kinds of linear algebraic groups

An algebraic group $\mathcal{G}$ is connected if the underlying variety is connected. This does not imply that $\mathcal{G}(F)$ is connected wrt. the topology coming from F.

Example: $GL_n(\mathbb{R})$ is connected (can't separate "$\det \gtrless 0$" using polynomials)

$$O_n(F) = SO_n(F) \amalg \{M \in O_n(F) : \det M = -1\} \text{ is disconnected.}$$

$\mathcal{G}$ is (almost) simple if $\mathcal{G}$, $\{\mathbb{1}\}$ are the only (connected) normal algebraic subgroups of $\mathcal{G}$ ($\mathcal{G}$ should be connected and noncommutative).

Examples: SL_n, SO_n, Sp_{2n}

$\mathcal{G}$ is semisimple if $\mathcal{G}$ is an almost direct product of simple subgroups of $\mathcal{G}$. This means that the product map $\prod_i \mathcal{G}_i \to \mathcal{G}$ is surjective and has finite kernel.

Def: $\mathcal{G}$ is <u>unipotent</u> if it is isomorphic to an algebraic subgroup of $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$.

(so $g-1$ is nilpotent $\forall g \in \mathcal{G}(F)$)

Example: $\mathcal{G}_a(F) = F = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\}$

Def: A linear algebraic group $\mathcal{G}$ is <u>reductive</u> if $\{1\}$ is the only connected, normal, unipotent, algebraic subgroup of $\mathcal{G}$.

Examples: GL_n, all semisimple groups.

Alternative characterization: $\mathcal{G}$ is reductive $\iff$ $[\mathcal{G},\mathcal{G}]$ is semisimple and $\mathcal{G}$ is the almost direct product of $[\mathcal{G},\mathcal{G}]$ and the center $Z(\mathcal{G})$.

From the representation theoretic point of view, reductive groups are the most interesting ones.

Def: An <u>algebraic torus</u> is a linear algebraic group which is diagonalizable (over $\bar{F}$)

Examples: $(\mathcal{G}_m)^n$, $\quad SO_2(F) = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x,y \in F, x^2+y^2=1 \right\}$

$$= \left\{ \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} : x,y \in F, x^2+y^2=1 \right\}$$

$$SO_2(\bar{F}) \doteq \left\{ \begin{pmatrix} x+\sqrt{-1}y & 0 \\ 0 & x-\sqrt{-1}y \end{pmatrix} : \begin{array}{c}(x+\sqrt{-1}y)(x-\sqrt{-1}y)=1, \\ x,y \in \bar{F}\end{array} \right\}$$

$$\Rightarrow \quad \begin{array}{ccc} SO_2(\bar{F}) & \longrightarrow & \bar{F}^* \\ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} & \longmapsto & x+\sqrt{-1}y \end{array} \qquad \text{Isomorphism}$$

Def: A torus $\mathcal{T}(F)$ is <u>F-split</u> if $\mathcal{T}(F) \cong \mathcal{G}_m(F)^{\dim \mathcal{T}}$ as isomorphism of algebraic groups over F. That means $\mathcal{T}(F) \to \mathcal{G}_m(F)^{\dim \mathcal{T}}$ given by polynomials w/ coefficients in F.

Example: $SO_2(F)$ is F-split iff char $F \neq 2$ and $\sqrt{-1} \in F$.

Def: An <u>F-rational character</u> of $\mathcal{G}$ is a homomorphism of algebraic groups $\mathcal{X}: \mathcal{G} \to \mathcal{G}_m$ which is defined over F.

$\mathcal{X}$ induces $\mathcal{X}: \mathcal{G}(F) \to F^*$.

The collection of F-rational characters of $\mathcal{G}$ is a group $\mathcal{X}^*(\mathcal{G}(F))$ w.r.t. pointwise multiplication of maps $\mathcal{G}(F) \to F^*$.

Lemma: Every F-rational character of $\mathcal{G}_m(F)$ is of the form $x \mapsto x^n$ for some $n \in \mathbb{N}$.

Proof: Let $\mathcal{X} \in \mathcal{X}^*(F^*)$. It induces an algebra homomorphism

$$\begin{array}{ccc} \mathcal{X}^*: \mathrm{Func}(\mathcal{G}_m(F)) & \longrightarrow & \mathrm{Func}(\mathcal{G}_m(F)) \\ \cong & & \cong \\ F[t,t^{-1}] & \longrightarrow & F[s,s^{-1}] \end{array}$$

$\mathcal{X}^*(t) \in F[s,s^{-1}]$ invertible $\Rightarrow \mathcal{X}^*(t) = cs^n$, $c \in F^*$, $n \in \mathbb{Z}$

$\Rightarrow \mathcal{X}(y) = cy^n \ \forall y \in F^*$, but $\mathcal{X}(y_1 y_2) = \mathcal{X}(y_1)\mathcal{X}(y_2)$, so $c = 1$ $\blacksquare$

Consequence: $\mathcal{X}^*(\mathcal{G}_m(F)) \cong \mathbb{Z}$.

$\mathcal{X}^*(T_1 \times T_2) \cong \mathcal{X}^*(T_1) \times \mathcal{X}^*(T_2) \Rightarrow \mathcal{X}^*(\mathcal{T}(F)) \cong \mathbb{Z}^{\dim \mathcal{T}}$ if $\mathcal{T}$ is F-split.

<u>Maximal Subtori (of reductive groups):</u>

Def: A <u>subtorus</u> of $\mathcal{G}(F)$ is an algebraic subgroup $\mathcal{T}(F)$ which is a torus (and $\mathcal{T}(F)$ is defined over F).

A <u>maximal F-split subtorus</u> is a subtorus which is F-split and is maximal for these properties.

Example: $SO_3(\mathbb{R})$: $\left\{ \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} : x^2+y^2 = 1 \right\}$ maximal subtorus of $SO_3(\mathbb{R})$ but not $\mathbb{R}$-split.

$\{1\}$ is a maximal $\mathbb{R}$-split subtorus of $SO_3(\mathbb{R})$.

M. Solleveld : p-adics IV

Thm (Borel–Tits: "Groupes réductifs"):

Let $\mathcal{G}$ be a connected linear algebraic group defined over F. Then any two maximal F-split subtori of $\mathcal{G}$ are conjugate in $\mathcal{G}(F)$.

From now on $\mathcal{G}$ will be connected and reductive, defined over F. We fix a maximal F-split subtorus $S = S(F)$ in $G = \mathcal{G}(F)$.

Def: $\mathcal{G}$ is F-split if S is a maximal torus in $\mathcal{G}$.

Examples: GL_n, SL_n are split for any F.
SO_n is not $\mathbb{R}$-split.

Def: The Weyl group $W(G,S) = N_G(S)/Z_G(S)$ (finite group) acts on S by conjugation.

Example: $G = SL_n(F)$, $S = \{\text{diagonal matrices in } G\} = Z_G(S)$.
$N_G(S) = \{\text{monomial matrices in } G\}$
$W(G,S) \cong S_n$ (symmetric group)

Roots of (G,S): S acts on $Lie(G)$ via the adjoint representation of G.

$$\Longrightarrow Lie(G) = \bigoplus_{\alpha \in \mathcal{X}^*(S)} V_\alpha, \quad V_\alpha := \{X \in Lie(G) : AXA^{-1} = \alpha(A)X \ \forall A \in S\}$$

The sum is direct, because it is an algebraic action of a torus.
$\alpha \in \mathcal{X}^*(S) \setminus \{0\}$ s.t. $V_\alpha \neq \{0\}$ are the roots of (G,S). $R(G,S) :=$ set of roots.
$V_0 = Lie(Z_G(S))$

Example: $G = SL_n(F)$, roots $\alpha_{ij}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = x_i x_j^{-1}$ $(i \neq j)$

$$V_{\alpha_{ij}} = F e_{ij} = Lie\left\{\begin{pmatrix} 1 & & y \\ & \ddots & \\ & & 1 \end{pmatrix} : y \in F\right\} =: Lie(U_{\alpha_{ij}}).$$

V_0 = diagonal matrices w/ trace 0 = $Lie(S)$.

Theorem (Borel–Tits): $V = \mathcal{X}^*(S/S \cap Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$ real vector space w/ $W(G,S)$-action. We endow it w/ a $W(G,S)$-invariant inner product. Then $R(G,S)$ is an integral root system in V w/ Weyl group $W(G,S)$.
If G is F-split $\Rightarrow$ $R(G,S)$ is reduced, i.e. $\mathbb{R}\alpha \cap R(G,S) = \{\alpha, -\alpha\}$.

<u>Axioms of a root system $\mathcal{R}$ in V:</u>

- $\mathcal{R}$ spans V
- $\mathcal{R}$ finite, $0 \notin \mathcal{R}$.
- $S_\alpha(\mathcal{R}) = \mathcal{R}$ where $S_\alpha : V \to V \quad \forall \alpha \in \mathcal{R}$, $v \longmapsto v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha,\alpha\rangle}\alpha$
- $2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z} \quad \forall \alpha,\beta \in \mathcal{R}$.

<u>Example:</u> $G = U_3(E/F, J) := \{ A \in GL_3(E) : A^t J \sigma(A) = J \}$, $J = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0\end{pmatrix}$

$Gal(E/F) = \{ id_E, \sigma \}$, E/F Galois extension of degree 2, $char(E) \neq 2$.

diagonal matrices in G: $\left\{ \begin{pmatrix} a&&\\&b&\\&&c\end{pmatrix} : a,b,c \in F, \begin{pmatrix} a&&\\&b&\\&&c\end{pmatrix} J \begin{pmatrix} \sigma(a)&&\\&\sigma(b)&\\&&\sigma(c)\end{pmatrix} = J \right\}$,

$\Rightarrow a\sigma(c) = 1 = c\,\sigma(a)$, $b\sigma(b) = 1$.

the b's do not give anything F-split (For $F = \mathbb{R}$ $b\sigma(b) = 1 \Rightarrow |b| = 1 \Rightarrow$ compact)

$S = \left\{ \begin{pmatrix} a&&\\&1&\\&&\sigma(a)^{-1}\end{pmatrix} : a \in E^* \right\}$ maximal F-split torus.

Claim: $\mathcal{T}(F) = \{ b \in E : b\sigma(b) = 1 \}$ not F-split

<u>Proof:</u> $E = F(\sqrt{t})$, $\sigma(a + b\sqrt{t}) = a - b\sqrt{t} \Rightarrow \mathcal{T}(F) = \{ (a,b) \in F^2 : (a+b\sqrt{t})(a-b\sqrt{t}) = 1 \}$

$\mathcal{T}(E) = \{ (c,d) \in E^2 : cd = 1 \} = \mathbb{G}_m(E)$

$\mathcal{X}^*(\mathcal{T}(E)) \cong \mathbb{Z}$, $\mathcal{X}_n : (c,d) \longmapsto c^n$

restrict to $\mathcal{T}(F)$: $\mathcal{X}_n : (a,b) \longmapsto (a + b\sqrt{t})^n \in E$ F-valued $\Leftrightarrow n = 0$.

$\Rightarrow \mathcal{X}^*(\mathcal{T}(F)) = \{0\} \Rightarrow$ not F-split $\square$

<u>roots:</u> $\alpha : \begin{pmatrix} a&&\\&1&\\&&\sigma(a)^{-1}\end{pmatrix} \longmapsto a$, $V_\alpha = \left\{ \begin{pmatrix} 0&x&0\\0&0&z\\0&0&0\end{pmatrix} : x,z \in E \right\} \cap Lie(G)$

$V_{2\alpha} = \left\{ \begin{pmatrix} 0&0&y\\0&0&0\\0&0&0\end{pmatrix} : y \in E \right\} \cap Lie(G)$

$\mathcal{R}(G,S) = \{ \alpha, -\alpha, 2\alpha, -2\alpha \}$, $(n\alpha)(M) := \alpha(M)^n$.

For which $x,y,z \in E$ does $\begin{pmatrix} 1&x&y\\0&1&z\\0&0&1\end{pmatrix}$ lie in $G = U_3$?

$$\begin{pmatrix} 0&0&1\\0&1&0\\1&0&0\end{pmatrix} \overset{!}{=} \begin{pmatrix} 1&0&0\\x&1&0\\y&z&1\end{pmatrix} \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0\end{pmatrix} \begin{pmatrix} 1&\sigma(x)&\sigma(y)\\0&1&\sigma(z)\\0&0&1\end{pmatrix} = \begin{pmatrix} 0&0&1\\0&1&x+\sigma(z)\\1&z+\sigma(x)&y+\sigma(y)+z\sigma(z)\end{pmatrix}$$

$\Rightarrow z = -\sigma(x)$, $y + \sigma(y) + x\sigma(x) = 0 \Rightarrow \left\{ \begin{pmatrix} 1&x&y\\&1&-\sigma(x)\\&&1\end{pmatrix}, x,y \in E : y + \sigma(y) + x\sigma(x) = 0 \right\}$

$U_{2\alpha} = \left\{ \begin{pmatrix} 0&0&y\\&0&\\&&0\end{pmatrix} : y \in E ; y + \sigma(y) = 0 \right\}$, $\underset{\supset U_{2\alpha}}{U_\alpha} = \left\{ \begin{pmatrix} 0&x&y\\0&0&-\sigma(x)\\0&0&0\end{pmatrix} : x \in E, \underbrace{x\sigma(x) + y + \sigma(y)}_{=0} \right\}$

$\dim_F U_{2\alpha} = 1$, $\dim_F U_\alpha = 3$.

Theorem: Let $\alpha \in R(G,S)$.

a) $\exists$ canonical connected unipotent F-algebraic subgroup U_α w/

$$\mathrm{Lie}(U_\alpha) = \begin{cases} V_\alpha & , 2\alpha \notin R(G,S) \\ V_\alpha + V_{2\alpha} & , 2\alpha \in R(G,S) \end{cases}$$

b) $Z_G(S)$ and the U_α, $\alpha \in R(G,S)$ generate G.

c) If G is F-split $\rightarrow U_\alpha \cong F^* = \mathbb{G}_m(F)$ (that it is 1-dim. is nontrivial). In the non-split case, the dimension of V_α can be large.

Parabolic subgroups: Let Δ be a basis of $R(G,S)$ $\longrightarrow$ $\underset{\substack{\| \\ \mathbb{R}_{\geq 0}\Delta \cap R(G,S)}}{R^+(G,S)}$, $R^-(G,S)$ positive/negative roots

For $\Omega \subseteq \Delta$, let P_Ω be the group generated by $Z_G(S)$ and the U_α w/ $\alpha \in R^+(G,S) \cup (\mathbb{R}\Omega \cap R^-(G,S))$

Def: The groups P_Ω are called the <u>standard parabolic subgroups</u>.

A general <u>parabolic subgroup</u> P is conjugate to some P_Ω.

G/P is a complete (projective) variety. This characterizes the parabolic subgroups.

Example: $G = SL_3(F)$, $P_\emptyset = \left\{\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}\right\}$ $\quad P_{\{\alpha_{12},\alpha_{23}\}} = G$.

$P_{\alpha_{12}} = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}\right\}$

$P_{\alpha_{23}} = \left\{\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}\right\}$

In general, the minimal parabolic subgroup does not have to be solvable. ("non-quasi-split groups"). Examples are more complicated ($\Leftrightarrow Z_G(S)$ not a torus)

M. Solleveld : p-adics V

Assumptions:
- F local nonarchimedean field w/ discrete valuation $v : F \to \mathbb{Z} \cup \{\infty\}$
- $\mathcal{G}$ connected reductive algebraic group defined over F. $G = \mathcal{G}(F)$.

Filtrations of the root subgroups:

Example: $GL_3(F)$ $\quad U_{\alpha_{13}} = \{ \underbrace{\begin{pmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{pmatrix}}_{=: u_{\alpha_{13}}(x)} : x \in F \}$

For $k \in \mathbb{R}$ $\quad U_{\alpha,k} := \{ u_\alpha(x) : v(x) \geq k \}$ compact open subgroup of U_α.

$\bigcup_{k \in \mathbb{R}} U_{\alpha,k} = U_\alpha \quad , \quad \bigcap_{k \in \mathbb{R}} U_{\alpha,k} = \{1\}$.

maximal torus S, $\quad S_r := \{ \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \in S : v(x_i) - 1 \geq r \ \forall i \}$, $r \in \mathbb{R}_{\geq 0}$

$S_r := S$, $r \in \mathbb{R}_{<0}$.

$n_{\alpha_{13}} := \underbrace{\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}}_{= u_{\alpha_{13}}(1)} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{pmatrix}}_{u_{-\alpha_{13}}(-1)} \underbrace{\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}}_{u_{\alpha_{13}}(1)} = \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix} \in N_G(S)$

The image of n_α in $W(G,S) = N_G(S) / Z_G(S)$ is the reflection s_α.

Theorem (Chevalley): Let $\mathcal{G}$ be F-split. There exist isomorphisms $u_\alpha : F \to U_\alpha$ for $\alpha \in R(G,S)$ w/ properties a) and b).

Put
- $U_{\alpha,k} = u_\alpha(v^{-1}[k,\infty])$, $k \in \mathbb{R}$.
- $U_{0,r} = S_r = \{ s \in S : v(\chi(s) - 1) \geq r \ \forall \chi \in X^*(S) \}$, $r \in \mathbb{R}_{\geq 0}$.
  $S_r = S$, $r < 0$.
- $n_\alpha := u_\alpha(1)\, u_{-\alpha}(-1)\, u_\alpha(1)$

a) For $\alpha, \beta \in R(G,S) \cup \{0\}$, $[U_{\alpha,r}, U_{\beta,k}] \subseteq \langle \bigcup_{\substack{n\alpha + m\beta \in R(G,S) \cup \{0\} \\ n,m \in \mathbb{Z}_{>0}}} U_{n\alpha + m\beta, nr + mk} \rangle$

group generated by these subgroups

b) $n_\alpha \in N_G(S)$, image of n_α in $W(G,S)$ is s_α,
$n_\alpha u_{-\alpha}(x) n_\alpha^{-1} = u_\alpha(-x)$.

This is contained in Chevalley's proof that $\mathcal{G}$ can be defined over $\mathbb{Z}$ ($\mathcal{G}$ split, reductive)

Bruhat and Tits interpreted this as " G has a prolonged valuated root datum". (filtration of centralizer)
This means in particular that U_α is filtered by compact open subgroups $U_{\alpha,k}$, $k \in \mathbb{R}$, and $Z_G(S)$ is filtered by compact open subgroups $U_{0,r} = S_r$, $r \geq 0$, such that a) holds. The assertion of b) has to be refined in general.

<u>Thm (Bruhat + Tits, "Groupes reductifs sur un corps local")</u>

Every connected reductive p-adic group has a prolonged valuated root datum.

This is what one needs to construct the affine building of G.

<u>Example:</u> E/F Galois extension of degree 2, $v_E(y) = \frac{v_F(y\sigma(y))}{2} \in \frac{1}{2}\mathbb{Z}$

$$G = U_3(E/F, J), \quad J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$
$$\Big| \; U_{2,1}(F)$$

$$S = \left\{ \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} : a \in F \right\}, \quad U_{2\alpha,k} = \left\{ \begin{pmatrix} 1 & & y \\ & 1 & \\ & & 1 \end{pmatrix} : y \in E, \; y + \sigma(y) = 0, \; v(y) \geq k \right\}.$$

$$U_{\alpha,k} = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & -\sigma(x) \\ & & 1 \end{pmatrix} : x, y \in E, \; y + \sigma(y) + x\sigma(x) = 0, \; v(x) \geq k, \; v(y) \geq 2k \right\}$$

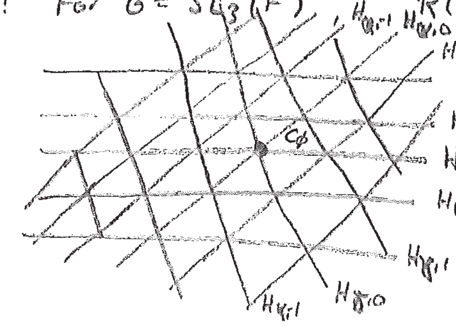
<u>The standard apartment of the affine building:</u>

$$X = \mathcal{X}^*(S/S \cap Z(G)), \quad A_S = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R})$$

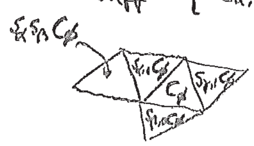
For $\alpha \in R(G,S)$ let $\Gamma_\alpha := \{k \in \mathbb{R} : U_{\alpha,r}$ jumps at $k = r\}$. This is a discrete subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. For $w \in W(G,S)$ $\Gamma_\alpha = \Gamma_{w\alpha}$.

<u>Def:</u> A wall in A_S is a hyperplane of the form $H_{\alpha,k} = \{y \in A_S : \langle y, \alpha \rangle = k\}$, $\alpha \in R(G,S)$, $k \in \Gamma_\alpha$.

These make A_S into a polysimplicial complex. polysimplicial = $\prod$ simplicial
chambers = connected components of $A_S \setminus \bigcup_{\alpha,k} H_{\alpha,k}$.
<u>Example!</u> For $G = SL_3(F)$, $R(G,S) =$



$S_{aff} = \{ s_{\alpha,0}, s_{\beta,0}, s_{\gamma,1} \}$



We fix a chamber $C_\emptyset$ w/ $\overline{C_\emptyset} \ni 0$. $S_{\alpha,k}$:= affine reflection in $H_{\alpha,k}$.

W_{aff} := subgroup of $A_S \rtimes W(G,S)$ generated by $S_{\alpha,k}$ $(k \in \Gamma_\alpha)$

S_{aff} := $\{ S_{\alpha,k} : H_{\alpha,k}$ is a wall of $C_\emptyset \}$

<u>Thm (Bourbaki):</u> a) (W_{aff}, S_{aff}) is a Coxeter system.

b) A_S is the associated Coxeter complex, i.e.

1.) W_{aff} acts simply transitively on the set of chambers.

2.) The neighbors of $w\,C_\emptyset$ are $\{ w s C_\emptyset : s \in S_{aff} \}$.

<u>The action of $N_G(S)$ on A_S:</u>

<u>Def</u>: $\nu: S \to A_S$, $\langle \nu(s), \chi \rangle := -v(\chi(s))$, $\chi \in \mathcal{X}^*(S/_{Z(G)\cap S})$.

[In the above example for $s = \begin{pmatrix} \pi & & \\ & \pi^{-1} & \\ & & 1 \end{pmatrix}$, $v(\pi) = 1 \to v(\alpha(s)) = v(\pi^2) = 2$, $v(\beta(s)) = v(\pi^{-1}) = -1$

$\to \nu(s) = \underset{(\text{coroot})}{-\alpha^\vee} := -\frac{2\alpha}{\langle \alpha, \alpha \rangle}$]

ν can be extended to $\nu: Z_G(S) \to A_S$ and further to $\nu: N_G(S) \to A_S \rtimes W(G,S)$ such that

- ν induces $\mathrm{id}_{W(G,S)}$: $N_G(S)/_{Z_G(S)} \longrightarrow A_S \rtimes W(G,S)/A_S$
- $\ker \nu = Z(G) \cdot$(maximal compact subgroup of $Z_G(S)$)
- for G F-split : $\nu(n_\alpha) = S_{\alpha,0}$ $\left(\rightsquigarrow \nu(u_\alpha(x) u_{-\alpha}(-1/x) u_\alpha(x)) = S_{\alpha,-v(x)}\right)$

This determines ν uniquely (in the split case)

[In the example, $g = u_{\alpha_{12}}(x)\, u_{-\alpha_{12}}(-1/x)\, u_{\alpha_{12}}(x) = \begin{pmatrix} 0 & x & 0 \\ -1/x & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x & & \\ & 1/x & \\ & & 1 \end{pmatrix} n_\alpha$

$\nu(g) = \nu\left(\begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} S_{\alpha,0}\right) = (-v(x)\alpha^\vee, S_{\alpha,0}) \in A_S \rtimes W(G,S)$.

$= S_{\alpha,-v(x)}$]

$A_S \rtimes W(G,S)$ acts on A_S by $(a,w) \cdot y = a + w(y)$, so ν defines an action of $N_G(S)$ on A_S. $\mathcal{B}(G)$ will be $G \times A_S/\sim$.

[Reference: Tits, Corvallis proceedings, "Reductive groups over a local field"]

We need isotropy groups (in G) of points of A_S. We decree that the fixed points of $U_{\alpha,k}$, $k \in \Gamma_\alpha$, are $\{ y \in A_S : k \geq \alpha(y) \}$, a half space in A_S.

$G_y :=$ group generated by $N_G(S)_y$ and $\bigcup_{\alpha \in R(G,S)} U_{\alpha, -\alpha(y)}$, $y \in A_S$.

G_y is not larger than expected : $G_y \cap U_\alpha = U_{\alpha,-\alpha(y)}$.

$G_y / Z(G) \cap G_y$ is compact.

Bruhat, Tits : $G_y = (U^- \cap G_y)(N_G(S) \cap G_y)(U^+ \cap G_y)$ (as sets)

where $U^\pm$ are the groups generated by $\bigcup_{\alpha \in R^\pm(G,S)} U_\alpha$ (for any reasonable choice of positive roots)

<u>Example:</u> $SL_3(\mathbb{Q}_p)_0 = SL_3(\mathbb{Z}_p)$

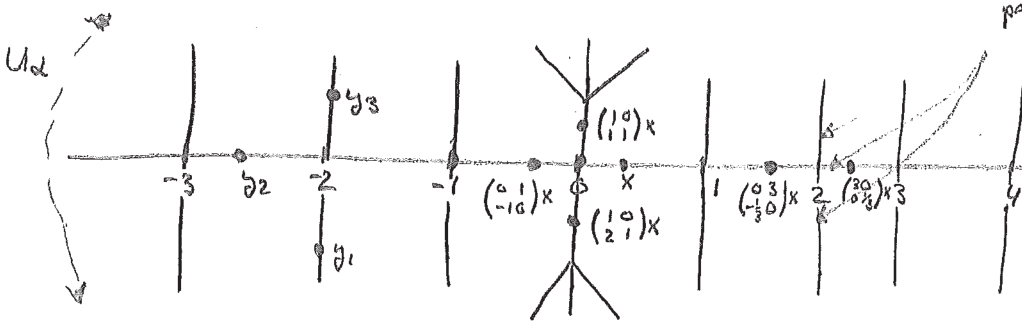
$$SL_3(\mathbb{Q}_p)_y = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap SL_3(\mathbb{Q}_p)$$ "Iwahori subgroup"

$y \in C_\emptyset$ interior point.

M. Solleveld : p-adics VI

Def: The Bruhat-Tits building of $G = \mathcal{G}(F)$ is $G \times A_S/\sim$, where $(g,x) \sim (h,y)$ iff $\exists n \in N_G(S)$, $n(x) = y$, i.e. $g^{-1}hn \in G_x$.

Example: $G = SL_2(\mathbb{Q}_3)$



parametrized by $G_{3/2} / G_{5/2} \cap ($

$G_{3/2} = \begin{pmatrix} \mathbb{Z}_3 & 3^{-1}\mathbb{Z}_3 \\ 3^2\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \cap SL_2(\mathbb{Q}_3$

$G_{3/2} \cap U_\alpha \cong U_{\alpha,-\frac{3}{2}} = \begin{pmatrix} 1 & \frac{1}{3}\mathbb{Z}_3 \\ 0 & 1 \end{pmatrix}$

$G_{3/2} \cap U_{-\alpha} \cong U_{-\alpha,\frac{3}{2}} = \begin{pmatrix} 1 & 0 \\ 3^2\mathbb{Z}_3 & 1 \end{pmatrix}$

$G_{5/2} = \begin{pmatrix} \mathbb{Z}_3 & 3^{-2}\mathbb{Z}_3 \\ 3^3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \cap SL$

$$G_{3/2} / G_{3/2} \cap G_{5/2} \cong \frac{\begin{pmatrix} 1 & 0 \\ 3^2\mathbb{Z}_3 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \\ 3^3\mathbb{Z}_3 & 1 \end{pmatrix}} = U_{-\alpha,2} / U_{-\alpha,3}$$

$$G_0 = SL_2(\mathbb{Z}_3), \quad G_x = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \cap SL_2(\mathbb{Q}_3)$$

$$G_0/G_x \cong U_{-\alpha,0}/U_{-\alpha,1} \cong \mathbb{Z}_3/3\mathbb{Z}_3$$

$$\{y_1, y_2, y_3\} \longleftrightarrow U_{\alpha,2}/U_{\alpha,3}$$

Properties:

1.) A_S embeds in $\mathcal{B}(G)$ via $y \longmapsto (1, y)$. (easy!)

2.) G acts on $\mathcal{B}(G)$ by $g(h,y) = (gh, y)$

3.) The isotropy group of $x \in A_S$ is indeed G_x.

4.) $x, y \in A_S$ in one G-orbit $\Rightarrow$ x, y in the same $N_G(S)$-orbit.

5.) $Z(G)$ acts trivially on $\mathcal{B}(G)$, because $Z(G) \subset G_x$ $\forall x \in A_S$.

6.) terminology: gA_S apartment
gC chamber (C chamber in A_S)
$g\sigma$ polysimplex in $\mathcal{B}(G)$ (σ polysimplex in A_S)

7.) $\mathcal{B}(G)$ polysimplicial complex, locally finite.

8.) The action of G preserves the polysimplicial structure.

9.) For any two polysimplices σ, τ in $\mathcal{B}(G)$, there is an apartment containing σ, τ.

In the picture: apartments $\cong$ lines extending indefinitely in both directions

$G_Y := \{g \in G : gy = y \quad \forall y \in Y\}$

<u>Lemma 1:</u> a) Let σ be a polysimplex in $\mathcal{B}(G) \Rightarrow G_\sigma$ acts transitively on the set of apartments containing σ.

b) Suppose $\sigma \subset A \Rightarrow G_\sigma A = \mathcal{B}(G)$. (A any apartment)

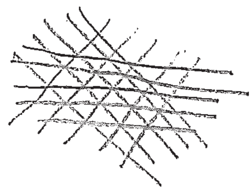
<u>Proof:</u> a) We may assume that $\sigma \subset A_S$. For generic $x \in \sigma$, $G_x = G_\sigma$ because G respects polysimplices. Suppose that an apartment A contains $\sigma \ni x$, $A = gA_S$ for some $g \in G$. Since $g^{-1}x \in A_S$ $\exists\, n \in N_G(S) : g^{-1}x = nx$, i.e. $gnx = x \Rightarrow gn \in G_x$, $A = gA_S = gnA_S$.

b) Follows from a) and 9.) $\square$

To understand the relation between $\mathcal{B}(G)$ and the picture visualizing it, observe that e.g.

$$\underset{\hat{A}_S}{SL_2(\mathbb{Z}_3)x_4} = \frac{SL_2(\mathbb{Z}_3)}{SL_2(\mathbb{Z}_3) \cap G_{x_4}} \cong \frac{U_{-\alpha,0}}{U_{-\alpha,4}}$$

$$G_{x_4} = \begin{pmatrix} \mathbb{Z}_3 & 3^{-4}\mathbb{Z}_3 \\ 3^4\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \cap SL_2(\mathbb{Q}_3)$$

For $G = SL_3(F)$, $A_S =$ 

along every wall $H_{\alpha,k}$ of A_S, $\mathcal{B}(SL_3, F)$ branches, the branches are parametrized by

$$U_{\alpha,k}/U_{\alpha,k+1} \cup \{*\} \qquad \frac{SL_2(\mathcal{O})}{B(\mathcal{O})}$$

$\mathcal{O} = \{x \in F : v(x) \geq 0\}$

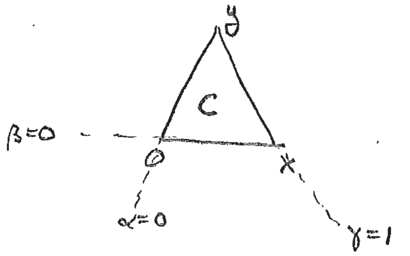
<u>Bruhat - Tits fixed point thm:</u> A subgroup of $G/Z(G)$ is compact $\Longleftrightarrow$ It fixes a point of $\mathcal{B}(G)$.

Corollary: There is a bijection $\{\text{vertices of } \mathcal{B}(G)\} \longleftrightarrow \left\{\begin{matrix}\text{maximal compact} \\ \text{subgroups of } G\end{matrix}\right\}$

$x \longleftrightarrow K_x =$ maximal compact subgroup of G_x $= (G_x \cap [G,G]) Z(G)$

For $G = SL_3(F)$ as above: $K_0 = SL_3(\mathcal{O})$

$$K_x = \begin{pmatrix} \mathcal{O} & \pi^{-1}\mathcal{O} & \pi^{-1}\mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL_3(F)$$ not conjugate to K_0.

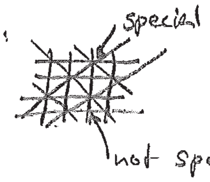


$\{K_0, K_x, K_y\}$ all conjugacy classes.

Def: A vertex $x \in A_S$ is special if $N_G(S)_x / Z_G(S)_x \cong W(G,S)$.

$0 \in A_S$ is special.

Example:



For $SL_3(F)$, all vertices are special.

Fix a basis Δ of $R(G,S)$. Positive cone $A_S^+ = \{x \in A_S : \langle x, \alpha\rangle \geq 0 \ \forall \alpha \in \Delta\}$.

$\nu : N_G(S) \longrightarrow A_S \rtimes W(G,S)$, $Z_G(S)^+ = \nu^{-1}(A_S^+)$.

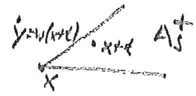
Thm: (Cartan decomposition)

a) Let $x \in A_S$ special vertex $\Rightarrow G = K_x Z_G(S)^+ K_x$

b) The natural map $Z_G(S)^+ / \ker \nu \longrightarrow K_x \backslash G / K_x$ is bijective.

Proof: a) Let $g \in G$, $gx \in \mathcal{B}(G)$. By Lemma ? $\exists k_1 \in K_x$ s.t. $\underbrace{k_1 g x}_{=y} \in A_S$

$\Rightarrow \exists n \in N_G(S) : n(x) = y$, $g^{-1}k_1^{-1} n \in G_x$



There are $t \in A_S^+ \cap \nu(Z_G(S))$, $w \in \frac{N_G(S)_x}{Z_G(S)_x}$ s.t. $y = w(x+t)$. Pick $z \in Z_G(S)$, $\tilde{w} \in N_G(S)_x$ w/ $\nu(z) = t$, $\nu(\tilde{w}) = w$.

$$\Rightarrow \underset{\overset{\|}{k_1 g x}}{y} = z\tilde{w}z \quad \Rightarrow \underbrace{g^{-1}k_1^{-1} z\tilde{w}}_{=k_2} \in G_x \quad \Rightarrow g = k_1^{-1} z \tilde{w} k_2^{-1} \in G_x Z_G(S)^+ G_x$$

Since $G_x \in Z(G) K_x$ and $Z(G) \subset Z_G(S)^+$, we can achieve $g \in K_x Z_G(S)^+ K_x$

b) Suppose that $g \in K_x z K_x \cap K_x z' K_x$. Show $z^{-1}z' \in \ker(\nu)$.

$gx \in K_x(zx) \cap K_x(z'x) \Rightarrow K_x(zx) = K_x(z'x)$

$\Rightarrow \exists\, n \in N_G(S)_x : nzx = z'x$. But $zx, z'x \in A_S^+$ and A_S^+ is a fundamental domain for the action of $N_G(S)_x$ on A_S

$\Rightarrow zx = z'x \Rightarrow z \ker(\nu) = z' \ker(\nu)$ $\quad (\nu(z) = \nu(z'))$. $\boxtimes$

$W^e = N_G(S)/\ker(\nu)$, a finite extension W_{aff}.

Let C be a chamber in A_S.

<u>Thm</u>: (affine Bruhat decomposition) Suppose $Z(G)$ is compact.

a) $G = G_C N_G(S) G_C \underset{\substack{\text{(sloppily}\\ \text{written)}}}{\approx} G_C W^e G_C$

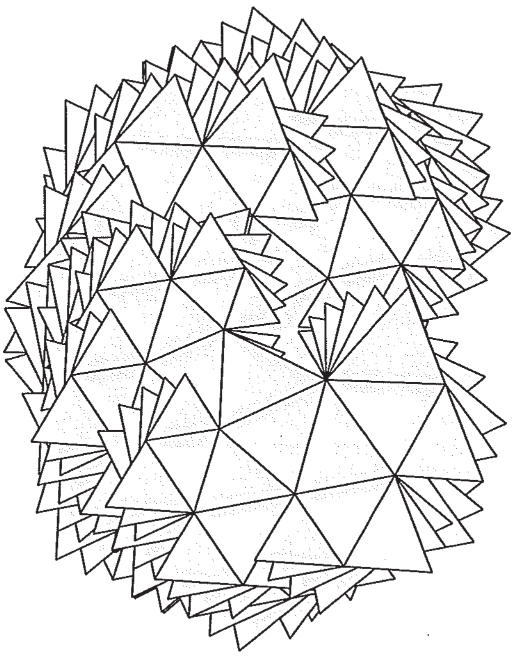
b) $W^e \longrightarrow G_C \backslash G / G_C$ bijection.

$\boxtimes$

<u>Proof</u>: like for the Cartan decomposition

<u>Example</u>: $G = SL_n(F)$, $G_C = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ & \ddots & \vdots \\ \pi\mathcal{O} & & \mathcal{O} \end{pmatrix} \cap SL_n(F)$.

There's also an Iwasawa decomposition. The main point in the proof is to show that $U^+ A_S = B(G)$.



Substantial fragment of a two-dimensional building

This emphasizes the visually chaotic nature of any two-dimensional representation of a thick building of dimension greater than one. One-dimensional affine buildings are simply trees, so can be rendered in a comprehensible and illuminating (as well as aesthetically interesting) manner. But in higher dimensions the thickness of the building is a very direct obstacle to creation of accurate two-dimensional models.

$U^+ :=$ group generated by $\bigcup_{\alpha \in R(G,S)^+} U_\alpha$.

Lemma 2 $U^+ \cdot A_S = \mathcal{B}(G)$.

"Proof." Let $y \in \mathcal{B}(G)$.
Choose a chamber $C \subset A_S$ "sufficiently" "deep" inside A_S^+.
$\exists\, x \in A_S$ such that $y \in G_C \cdot x$ and $x + A_S^+ \supset C$.
$= g \cdot x$
by Lemma 1.b.

$$g \in \underset{\substack{\| \\ G_a}}{G_C} = (G_C \cap U^+)\underbrace{(G_C \cap N_G(S))}_{= \ker \nu}(G_C \cap U^-)$$

for suitable $a \in C$

since $C \subset A_S$ is open

fixes x since
$\forall \alpha \in R^-(G,S)$; $a \in C$
$-\alpha(x) \leq -\alpha(a)$

$$\Rightarrow \underset{\substack{\| \\ U_\alpha \cap G_x}}{U_{\alpha, -\alpha(x)}} \supset \underset{\substack{\| \\ U_\alpha \cap G_C}}{U_{\alpha, -\alpha(a)}}$$

So $y = g\, x = u\, x$ for some $u \in G_C \cap U^+$. $\square$

Theorem (Iwasawa decomposition)
Let $x \in A_S$ be special and let $P_\emptyset = U^+ Z_G(S)$ be the standard minimal parabolic subgroup of G.

a) $G = P_\emptyset K_x = K_x P_\emptyset$

b) For any parabolic P and any good maximal compact subgroup K : $G = PK = KP$.

Proof a) Let $g \in G$. By Lemma 2 $\exists\, u \in U^+$, $y \in A_S$ such that $gx = uy$
$\Rightarrow \exists\, n \in N_G(S)$: $n(x) = y$, $g^{-1}un \in G_x$
$= h$
Since x is special, we can write $nh^{-1} = zk$ with $z \in Z_G(S)$ and $k \in K_x$ $\Rightarrow g = unh = uzk \in P_\emptyset K_x$.

b) By definition $\exists g \in G$: $gPg^{-1} \supset P_\emptyset$.

By a) we can write $g = pk$ with $p \in P_\emptyset$, $k \in K_X$

$$PK_X \supset g^{-1}P_\emptyset g K_X = k^{-1}p^{-1}P_\emptyset pk K_X = k^{-1}P_\emptyset K_X = k^{-1}G$$

$$\Longrightarrow PK_X = G = K_X P$$

Moreover K is conjugate to some K_Y with $Y \in A_S$ special.

$K = hK_Y h^{-1} \Longrightarrow$

$$PK = PhK_Y h^{-1} = h(h^{-1}Ph)K_Y h^{-1} = hGh^{-1} = G. \quad \square$$