The fundamental group of a II_1 factor

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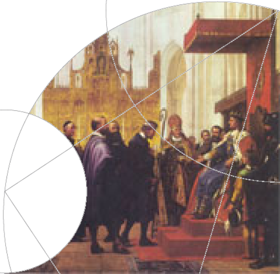
september 15, 2011



Type II_1 factors

Definition

A von Neumann algebra M is a (separable) type II_1 factor if

- M has trivial centre: $\mathcal{Z}(M) = \mathbb{C}$
- M has a trace $\tau : M \to \mathbb{C}$
  - τ is a state
  - $\tau(xy) = \tau(yx)$
  - faithful, normal
- we require that M acts on a separable hilbert space

Examples

- Group von Neumann algebra $L(\Gamma)$ of an ICC group
- $L^\infty(X) \rtimes \Gamma$ of a free, p.m.p. ergodic action

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When do different constructions give different II_1 factors?

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When do different constructions give different II_1 factors?

- $R = \mathrm{L}(\Gamma)$ is the same for all ICC amenable groups. (Connes, 1976)
  - $R = \mathrm{L}(\Gamma)$ if and only if Γ is ICC amenable.
- $R \neq \mathrm{L}(\mathbb{F}_n)$ for any $1 < n \leq \infty$ (Murray–von Neumann, 1943)
- Is $\mathrm{L}(\mathbb{F}_n) = \mathrm{L}(\mathbb{F}_m)$ if $n \neq m$? (Major open problem)

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Definition and history

Definition (Murray–von Neumann, 1943)

- Let M be a II_1 factor
  - if $p \in M$ is a projection, then pMp is again a II_1 factor
  - up to isomorphism, it only depends on $\tau(p)$
- The fundamental group is
$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid pMp \cong qMq\} \subset \mathbb{R}_+^\times .$$
- this is a subgroup of $\mathbb{R}_+^\times$

Examples

- $\mathcal{F}(R) = \mathbb{R}_+^\times$ (Murray–von Neumann, 1943)
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Which subgroups of $\mathbb{R}_+^\times$ are fundamental groups?

- only $\mathbb{R}_+^\times$ itself?
- only $\mathbb{R}_+^\times$ and countable?

Examples

- $\mathcal{F}(\mathrm{L}(\Gamma))$ is countable if Γ has ICC, (T) (Connes, 1980)
- $\mathcal{F}(\mathrm{L}(\mathrm{SL}_2\,\mathbb{Z} \ltimes \mathbb{Z}^2)) = \{1\}$ (Popa, 2002)
- $\mathcal{F}(M)$ can be any countable subgroup of $\mathbb{R}_+^\times$ (Popa, 2003)
- many uncountable groups are $\mathcal{F}(M)$ (Popa–Vaes, 2008)
- My result (D., 2010)

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- $\mathcal{F}(\mathrm{L}(\Gamma))$ is countable if Γ has ICC, (T) (Connes, 1980)
  - we can not compute $\mathcal{F}(\mathrm{L}(\Gamma))$ for any ICC property (T) group
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  - alternative constructions: Ioana-Peterson-Popa, Houdayer
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  - explicit construction
  - potentially larger class of groups

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The construction

- We begin with two actions and a quotient
  - "generic action": $\Lambda \curvearrowright (Y,\nu)$: ergodic, inf. m. p.
  - "specific action": $\Gamma \curvearrowright (X,\mu)$: free, ergodic, p.m.p.
  - quotient $\pi : \Gamma \to \Lambda$.
- $\Gamma \curvearrowright X \times Y$ by $g(x,y) = (gx, \pi(g)y)$
  - free, i.m.p.
  - ergodic if $\ker \pi \curvearrowright (X,\mu)$ is ergodic
- $M = L^\infty(X \times Y) \rtimes \Gamma$: a II_∞ factor
  - every isomorphism $\psi : M \to M$ scales Tr by $\mathrm{mod}(\psi)$.
  - if $\mathrm{Tr}(p) < \infty$, then pMp is a II_1 factor.
  - $\mathcal{F}(pMp) = \mathrm{mod}(\mathrm{Aut}(M))$
- $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}(Y,\nu)}(\Lambda)) \subset \mathrm{mod}(\mathrm{Aut}(M))$
  - if $\Delta \in \mathrm{Centr}_{\mathrm{Aut}(Y,\nu)}(\Lambda)$, then $\mathrm{id} \times \Delta$ commutes with Γ
  - so $\psi(au_g) = (\mathrm{id} \times \Delta)_*(a)u_g$ defines an automorphism of M
- Strong conditions on $\Gamma \curvearrowright (X,\mu)$: $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}(Y,\nu)}(\Lambda)) = \mathrm{mod}(\mathrm{Aut}(M))$
  - Popa–Vaes conditions: no explicit examples + Λ amenable + $\Lambda \curvearrowright Y$ free
  - my set of conditions: explicit examples + all Λ + non-free actions

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Example

There are examples with $\pi : \Gamma \to \Lambda = \mathbb{F}_\infty$.

Corollary

For any ergodic, i.m.p. $\alpha : \Lambda \curvearrowright (Y, \nu)$ of **any** group, there is a II_1 factor M_α with

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For any closed subgroup $\mathcal{G} \subset \mathrm{Aut}_\nu(Y)$ that acts ergodically on Y, there is a type II_1 factor $M_\mathcal{G}$ such that

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Generalization

- We replace (Y,ν) by a II_∞ factor (B,Tr).
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Examples

Generalization

- We replace (Y, ν) by a II_∞ factor (B, Tr).
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For any trace-preserving outer action $\alpha : \Lambda \to \mathrm{Out}_{\mathrm{Tr}}(B)$ of any countable group Λ, there is a II_1 factor M_α with

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