Universal coefficient theorems for C^*-algebras over finite topological spaces

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Classification of simple nuclear purely infinite C^*-algebras

Theorem (Rosenberg-Schochet 1987)

Let A and B be separable C^*-algebras.
If A belongs to the bootstrap category, then there is a short exact sequence of $\mathbb{Z}/2$-graded Abelian groups

$$\mathrm{Ext}^1(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)) \rightarrowtail \mathrm{KK}_*(A, B) \twoheadrightarrow \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)).$$

Corollary

If A and B are in the bootstrap class then an isomorphism $\mathrm{K}_*(A) \cong \mathrm{K}_*(B)$ lifts to a KK-equivalence $A \simeq B$.

Theorem (Kirchberg-Phillips 2000)

A KK-equivalence between two stable, nuclear, separable, purely infinite, simple C^*-algebras lifts to a *-isomorphism.

Classification results for non-simple C^*-algebras

Theorem (Rørdam 1997)

Extensions of stable, nuclear, separable, purely infinite, simple C^*-algebras in the bootstrap class are classified by their six-term exact sequence.

Theorem (Restorff 2006)

Cuntz Krieger algebras whose adjacency matrices fulfill condition (II) are classified up to stable isomorphism by filtered K-theory.

Idea: Reprove these theorems using a 2-step classification via an equivariant KK-theory!

C^*-algebras over topological spaces

Throughout this talk, X denotes a finite connected T_0-space.

Definition

A C^*-algebra over X is a pair (A, ψ) consisting of a C^*-algebra A and a continuous map $\psi\colon \operatorname{Prim}(A) \to X$.

- An open subset $U \subseteq X$ gives a distinguished ideal $A(U)$ of A.
- A *-homomorphism $f\colon A \to B$ is X-equivariant if $f(A(U)) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$.
- $\rightsquigarrow$ Category $\mathfrak{C}^*\mathfrak{alg}(X)$ of C^*-algebras over X.

Equivariant bivariant K-theory

Eberhard Kirchberg constructed an X-equivariant version $\mathrm{KK}(X)$ of Kasparov's KK-theory.

Kasparov product

$$\mathrm{KK}_*(X; A, B) \otimes \mathrm{KK}_*(X; B, C) \to \mathrm{KK}_*(X; A, C)$$

Definition

Let $\mathfrak{KK}(X)$ be the category of separable C^*-algebras over X with $\mathrm{KK}_0(X; \textvisiblespace, \textvisiblespace)$ as morphism groups.

Theorem (Meyer-Nest)

The category $\mathfrak{KK}(X)$ is triangulated and has countable coproducts.

Equivariant bivariant K-theory

Theorem (Kirchberg 2000)

A $KK(X)$-equivalence between two stable, nuclear, separable, purely infinite, tight C^*-algebras over X lifts to an X-equivariant *-isomorphism.

Here a C^*-algebra (A, ψ) over X is called tight if $\psi\colon \mathrm{Prim}(A) \to X$ is homeomorphic.

Filtered K-theory

Vague Definition

The filtered K-theory $\mathrm{FK}(A)$ of a C^*-algebra over X comprises

- the K_*-groups of all quotients of distinguished ideals in A,
- all natural maps between these groups.

Alternative picture (Meyer-Nest)

$\mathrm{FK}(A) = \mathrm{KK}(X; \mathcal{R}_X, A)$ as module over the ring $\mathrm{KK}(X; \mathcal{R}_X, \mathcal{R}_X)$.

- Meyer-Nest also define an equivariant bootstrap class $\mathcal{B}(X) \subseteq \mathfrak{KK}(X)$.

UCT with homological condition

Using the universal property of FK and their machinery of homological algebra in triangulated categories, Meyer-Nest prove:

Theorem (Meyer-Nest)

Let $A, B \in \mathfrak{KK}(X)$. Suppose that $\mathrm{FK}(A)$ has a projective resolution of length 1 and that $A \in \mathcal{B}(X)$. Then there are natural short exact sequences

$$\mathrm{Ext}^1(\mathrm{FK}(A)[j+1], \mathrm{FK}(B)) \rightarrowtail \mathrm{KK}_j(X; A, B) \twoheadrightarrow \mathrm{Hom}(\mathrm{FK}(A)[j], \mathrm{FK}(B))$$

for $j \in \mathbb{Z}/2$.

Intermezzo: finite T_0-spaces as directed graphs

Let X be a finite set. There are the following bijections:

$$\{T_0\text{-topologies on } X\}$$

Alexandrov topology ↑ ↓ specialisation preorder

$$\{\text{partial orders on } X\}$$

reachability relation ↑ ↓ Hasse diagram

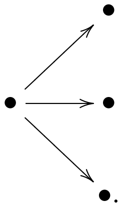
$$\{\text{transitively reduced directed acyclic graphs with vertex set } X\}$$

History

1.
   - $X = \bullet$
   - $\mathfrak{C}^*\mathfrak{alg}(X)$ = plain C^*-algebras
   - Universal Coefficient Theorem: Rosenberg-Schochet (1987)
2.
   - $X = \bullet \longrightarrow \bullet$
   - $\mathfrak{C}^*\mathfrak{alg}(X)$ = Extensions of C^*-algebras,
     FK = Six-term exact sequence
   - UCT: Bonkat (2002)
   - Reproves Rørdam's classification theorem!
3.
   - $X = \bullet \longrightarrow \bullet \longrightarrow \bullet$
   - UCT: Restorff (2008)
4.
   - $X = \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$
   - UCT: Meyer-Nest (2008)

A counterexample

At the same time, Meyer and Nest give a counterexample over the space

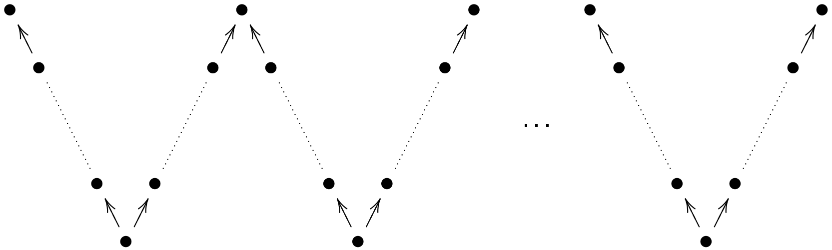


They construct:
$A, B \in \mathcal{B}(X)$ with $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ but $A \not\cong_{\mathrm{KK}(X)} B$.

Question

Which finite spaces allow for a UCT using filtered K-theory and which ones do not?

Accordion spaces



Accordion spaces

Theorem (B-Köhler)

The following statements are equivalent:

1. X *is an accordion space.*
2. *Let* A *and* B *be separable* C^**-algebras over* X.
   Suppose $A \in \mathcal{B}(X)$.
   Then there is a natural short exact UCT sequence

$$\mathrm{Ext}^1\big(\mathrm{FK}(A)[1], \mathrm{FK}(B)\big) \rightarrowtail \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}\big(\mathrm{FK}(A), \mathrm{FK}(B)\big).$$

3. *Let* $A, B \in \mathcal{B}(X)$. *Then* $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ *implies* $A \simeq_{\mathrm{KK}(X)} B$.

Measuring the failure of projective dimension 1

Theorem (B)

For $4 \begin{smallmatrix} \nearrow 3 \\ \rightarrow 2 \\ \searrow 1 \end{smallmatrix}$, $M = \mathrm{FK}(A)$ *has projective dimension at most 1 if and only if the homology of the following complex is free:*

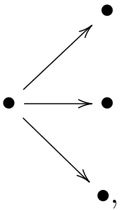
$$\bigoplus_{j=1}^{3} M(j4) \xrightarrow{\begin{pmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{pmatrix}} \bigoplus_{k=1}^{3} M(1234 \setminus k) \xrightarrow{(i\ i\ i)} M(1234).$$

Corollary (B)

There is a Cuntz-Krieger algebra with projective dimension 2 in filtered K-theory over its primitive ideal space.

Refining filtered K-theory

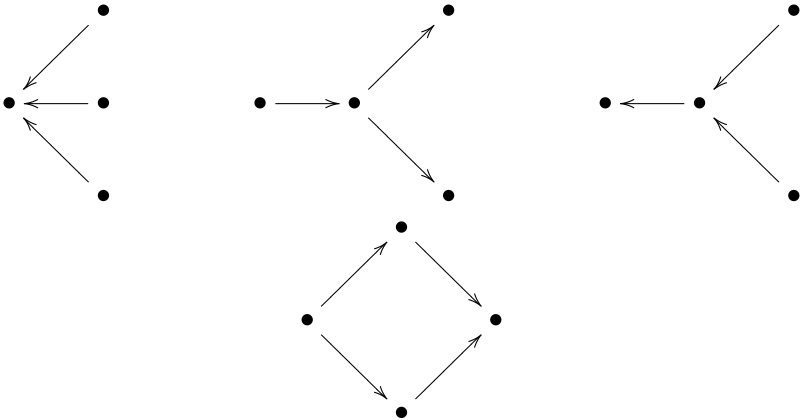
For the space



Meyer and Nest construct a refinement of filtered K-theory by adding one more invariant: the K-group of some pullback of distinguished subquotients of A. This invariant has a UCT!

Refining filtered K-theory

The same works for the following spaces (B):

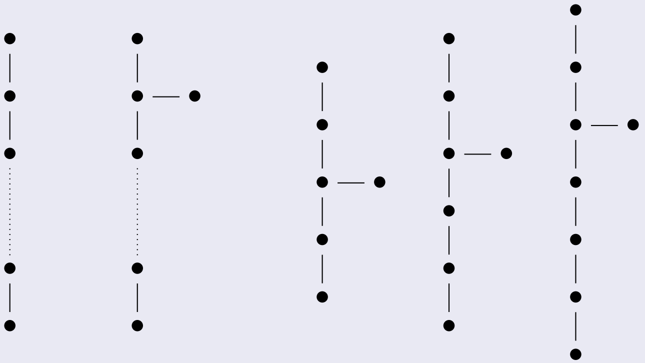


Question

Which finite spaces allow for a UCT
using a finite refinement of filtered K-theory?

Conjecture (Meyer, Katsura)

Let X be a tree. Then there is a UCT for a finite refinement of filtered K-theory if and only if the associated undirected graph of X is a simply laced Dynkin diagram:



Conjecture (B-Katsura)

Given two finite spaces X and Y with the same number of points, it "frequently" happens that we can construct adjunctions

$$\mathfrak{KK}(X) \longleftrightarrow \mathfrak{KK}(Y)$$

- *with large fixed subcategories,*
- *preserving the triangulated category structures,*
- *identifying $\mathcal{R}_X$ and $\mathcal{R}_Y$.*

This should be true:

- *if X is a tree and Y arises from X by reversing the direction of one arrow;*
- *for the two spaces*