On Polish Groups of Finite Type

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Joint work with
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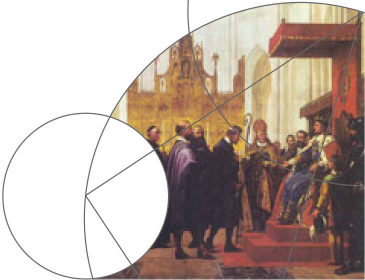

Motivation

My study interests: Polish groups related to operator algebras, $\mathcal{U}(M)$, $\text{Aut}(M)$, etc.

Definition

(1) A topological space is called *Polish* if it is separable and completely metrizable.

(2) A Polish group is a topological group whose topology is Polish.

- Structures of $\mathcal{U}(M)$, $\text{Aut}(M)$ are closely related to the structure of M (e.g. Connes' classification of injective factors)
- Such groups provide rich examples of (exotic) Polish Groups. (e.g. extremely amenable groups)

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Motivation

In the previous work we proved

Theorem

Let M be a finite von Neumann algebra(vNa). $G \subset \mathcal{U}(M)$ sot-closed subgroup. Then

$$\mathfrak{g} := \{A^* = -A; e^{tA} \in G, \ \forall t \in \mathbb{R}\}$$

is a complete topological Lie algebra w.r.t strong resolvent topology.

So if $A, B \in \mathfrak{g}$(possibly unbounded), $A + B$ and $AB - BA$ exist in $\mathfrak{g}$ and these operations are SRT-continuous.

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Problem

What kind of Polish group G can be embeddable into some $\mathcal{U}(M)$, M finite vNa? Examples of fintie type groups?

Definition (Popa'06)

A Polish group G is called of *finite type* if it is isomorphic onto a sot-closed subgroup of some $\mathcal{U}(M)$, M finite vNa.

We denote the above condition as $G \hookrightarrow \mathcal{U}(M)$.

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Embeddings into the unitary group of II_1 factor

Proposition (Popa, Haagerup-Winsløw)

Let G be a Polish Group. TFAE.

(1) *G is of finite type: $G \hookrightarrow \mathcal{U}(M)$, M finite vNa.*

(2) *$G \hookrightarrow \mathcal{U}(M)$, M separable II_1 factor.*

We first observe the known result for $G \hookrightarrow \mathcal{U}(\ell^2)$.

Embeddings into $\mathcal{U}(\ell^2)$

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OBS:
If $G \hookrightarrow \mathcal{U}(\ell^2)$. Suppose $g_n \to g$ in G. Then $\forall \xi \in \ell^2 \setminus \{0\}$,

$$\varphi_\xi(g_n) = \langle g_n\xi, \xi\rangle \to \langle g\xi, \xi\rangle = \varphi_\xi(g).$$

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Conversely, if $\varphi_\xi(g_n) \to \varphi_\xi(g)$ $\forall \xi$, then by polarization

$$\langle g_n\xi, \eta\rangle \to \langle g\xi, \eta\rangle, \quad \forall \xi, \eta.$$

so $g_n \to g$ weakly(=strongly).

Embeddings into $\mathcal{U}(\ell^2)$

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so $g_n \to g$ weakly(=strongly).
$\Rightarrow \{\varphi_\xi\}$ generates the topology of G.

Embeddings into $\mathcal{U}(\ell^2)$

Definition

A Polish Group G is called *unitarily representable* (UR) if $G \hookrightarrow \mathcal{U}(\ell^2)$.

The following result has been known to specialists, without explicitly mentioned until the work of Gao('03)

Theorem (Gao+ ??)

For a Polish group G. TFAE.

(1) *G is UR.*

(2) *$\exists f$ positive definite function on G which generates the topology of G: $g_n \to g \Leftrightarrow f(g_n) \to f(g)$.*

(3) *$\exists f$ positive definite function on G which separates each closed set $A(\not\ni 1)$ and 1: $\sup_{g\in A} |f(g)| < f(1)$*

Characterization of Finite Type Groups

Let G be a group. A function $f : G \to \mathbb{C}$ is called *invariant* if $f(g^{-1}xg) = f(x)$ holds $\forall g, x \in G$.

Theorem

Let G be a Polish Group. TFAE.

(1) *G is of finite type.*

(2) $\exists \{f_i\}_{i \in I}$ *conti. pos. def. invariant functions on G generating nbd basis of* 1*:* $g_n \to 1 \Leftrightarrow f_i(g_n) \to f_i(1), \forall i \in I$.

(3) $\exists f : G \to \mathbb{C}$ *conti. pos. def. inv, which generates nbd basis of 1.*

(4) $\exists f : G \to \mathbb{C}$ *conti. pos. def. inv. which separates each closed set $A (\not\ni 1)$ and* 1*:* $\sup_{g \in A} |f(g)| < f(1)$.

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Remark

Unlike UR case, f in (3) or (4) cannot generate the whole topology of G in general.

Another Characterization

Suppose $G \hookrightarrow \mathcal{U}(M)$, M II_1 factor with a normal faithful trace τ. Then G has a bi-invariant metric d:

$$d(u,v) := ||u-v||_2, u, v \in G.$$

Here, $||x||_2 := \tau(x^*x)^{\frac{1}{2}},\ x \in M$.

Definition

A topological group G is called a *SIN*-group if it admits a basic neighborhood system $\{V_i\}_{i\in I}$ of 1 which are invariant: $g^{-1}V_i g = V_i, \forall g \in G, \forall i \in I$.

Remark

a Polish group G is SIN iff G admits a compatible bi-invariant metric.

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Popa's Question

Sorin Popa asked the following question from the viewpoint of his cocycle superrigidity theory:

Problem (Popa('05))

If a Polish SIN group G is unitarily representable ($G \hookrightarrow \mathcal{U}(\ell^2)$), is it of fintie type?

We have partial answers to this problem.

Locally Compact Case

It iw known that for the case of locally compact groups, Popa's question has an affirmative answer (Kadison-Singer+ others)
We give a simpler proof using the characterization of finiteness.
Note: locally compact group is always unitarily representable (UR).

Theorem

A second countable locally compact group is of finite type iff it is a SIN-group.

Proof.

For each U compact invariant nbd at 1, define an pos. def. function

$$\varphi_U(g) := \langle \chi_U, \lambda(g)\chi_U \rangle = \mu(U \cap gU), \quad g \in G.$$

By the invariance of U and unimodular property of G, it is invariant: And then the family $\{\varphi_U\}_U$ generates a nbd basis of 1. □

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UR Amenable Groups

A Hausdorff topological group G is called *amenable* if it admits an (left) invariant mean:
$\exists m \in \mathrm{LUCB}(G)^*_+, m(1) = 1$ with $m(\lambda_g(f)) = m(f), \forall g \in G$.
Here, $\mathrm{LUCB}(G) :=$ the set of all left-uniformly continuous functions on G and $\lambda_g(f)(x) := f(g^{-1}x)$.

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Proof.

By UR, $\exists f$ pos def on G which generates the top of G. Define

$$\Psi_{x,f}(g) := f(g^{-1}xg), \quad x, g \in G.$$

Then one can show $\Psi_{x,f} \in$LUCB(G). Then

$$\psi_f(x) := m(\Psi_{x,f}), \quad x \in G$$

is conti. pos. def. inv on G and one can show that it separates closed set $A(\not\ni 1)$ and 1. □

Examples of Finite Type Polish Groups

Definition

Let M be a separable semifinite vNa with normal faithful semifinite trace τ. The group $\mathcal{U}(M)_2 := \{u \in \mathcal{U}(M);\ 1 - u \in L^2(M, \tau)\}$ is called a L^2-*unitary group* of M. It is equipped with a metric

$$d(u, v) := ||u - v||_2, u, v \in \mathcal{U}(M)_2.$$

Theorem

$(\mathcal{U}(M)_2, d)$ *is a Polish group of finite type.*

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Completeness of d: Let $\{u_n\}$ Cauchy seq in $\mathcal{U}(M)_2$.
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Finiteness: $f(u) := e^{-||u-1||_2^2}, \quad u \in \mathcal{U}(M)_2$ works. □

II_1 factor with Property (T)

Definition

II_1 factor (M, τ) has property (T) $\Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0$ and $\mathcal{F} \subset M$ finite s.t.
if $\varphi : M \to M$ is τ-preserving ucp map s.t. $||\varphi(x) - x||_2 < \delta$ $\forall x \in \mathcal{F}$, then $||\varphi(a) - a||_2 < \varepsilon ||a||\ \forall a \in M$.

Let (M, τ) sep. II_1 factor. Then $\mathrm{Aut}(M)$ with the topology of pointwise $|| \cdot ||_2$-convergence is a UR Polish SIN group.
Recently, the following result was communicated to us by Uffe Haagerup.

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Let M be a separable II_1 factor with property (T). Then $\mathrm{Aut}(M)$ is a Polish group of finite type.

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Permanence Properties

Operation	Finite Type?
Closed subgroup $H < G$	YES
Countable direct product $\prod_{n\geq 1} G_n$	YES
Quotient G/N	NO
Extension $1 \to N \to G \to K \to 1$	NO
Projective limit $\varprojlim G_n$	YES

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inductive limit? Gao-van den Dries Polish SIN-group?

Towards the Answer to Popa's Question

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$\exists X(\cdot), Y(\cdot)$ conti. one-para subgroups of G_0 s.t. the limit

$$\lim_{n\to\infty} \left\{ X\left(\frac{t}{n}\right) Y\left(\frac{t}{n}\right) \right\}^n$$

does NOT exist.

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does NOT exist.
But we showed('10): if G_0 is of finite type, the above limit always exists for $\forall$ one-para subgroups (Lie sum exists)!

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$G_0 = \overline{\mathbb{Q} * \mathbb{Q}}^{\delta}$ Gao-van den Dries group.

Question

Does $G_0 \hookrightarrow \mathcal{U}(\ell^2)$ hold?

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We are still working on the problem...