Radial Multipliers on Reduced Free Products

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Freenes

Definition

For $(\mathcal{A}_i) \subseteq (\mathcal{A}, \omega)$ unital C*-subalgebras we say $(\mathcal{A}_i)$ are free in $\mathcal{A}$ if

$$\forall n \in \mathbb{N} \quad \forall a_j \in \mathcal{A}_{i_j}, 1 \leq j \leq n, i_j \neq i_{j+1}, \omega(a_j) = 0$$

we have

$$\omega(a_1 a_2 \cdots a_n) = 0.$$

Reduced free product

Given unital C*-algebras $(\mathcal{A}_i, \omega_i)$ construct algebra $(\mathcal{A}, \omega)$ such that

- $(\mathcal{A}_i)$ free in $(\mathcal{A}, \omega)$
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Notation

$$(\mathcal{A}, \omega) = *_i(\mathcal{A}_i, \omega_i)$$

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Properties of reduced free product

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- $*_i C_r^*(G_i) = C_r^*(*_i G_i)$
- $*_i L(G_i) = L(*_i G_i)$
- Dense subset $A = \mathbb{C}1 \oplus \bigoplus_n \bigoplus_{i_1,\dots i_n, i_j \neq i_{j+1}} \mathring{\mathcal{A}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{A}}_{i_n}$ where $\mathring{\mathcal{A}}_i = \ker \omega_i$

Problem

Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(a_1 \dots a_n) = \phi(n) a_1 \dots a_n.$$

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- When is M_ϕ completely bounded?
- For which $\mathcal{A}_i$?
- For which ϕ?
- $\|M_\phi\|_{cb} = ?$

Class $\mathcal{C}$

Definition

Let $\mathcal{C}$ denote the set of functions $\phi : \mathbb{N}_0 \to \mathbb{C}$ for which the Hankel matrices

$$h = (\phi(i+j) - \phi(i+j+1))_{i,j\geq 0}$$
$$k = (\phi(i+j+1) - \phi(i+j+2))_{i,j\geq 0}$$

are of trace class and $c = \lim_{n\to\infty} \phi(n)$ exists.
For $\phi \in \mathcal{C}$ put

$$\|\phi\|_{\mathcal{C}} = \|h\|_1 + \|k\|_1 + |c|.$$

Known results

Theorem (Wysoczanski 1995)

Let $G = *_{i \in I} G_i$ and $\phi \in \mathcal{C}$ then $M_\phi : C^*_r(G) \to C^*_r(G)$ is welldefined and

$$\|M_\phi\|_{cb} \leq \|\phi\|_{\mathcal{C}}.$$

Theorem (Ricard-Xu 2006)

Let $\mathcal{A} = *_i \mathcal{A}_i$ and $\phi(n) = s^n$, $s \in (0,1)$ then $M_\phi : \mathcal{A} \to \mathcal{A}$ is welldefined and

$$\|M_\phi\|_{cb} \leq 1.$$

Our result

Theorem (Haagerup-M 2010)

Let $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ be the reduced free product of unital C-algebras $(\mathcal{A}_i)_{i \in I}$ with respect to states $(\omega_i)_{i \in I}$ for which the GNS-representation π_{ω_i} is faithful for all $i \in I$.*
If $\phi \in \mathcal{C}$, then there is an unique linear completely bounded map

$$M_\phi : \mathcal{A} \to \mathcal{A}$$

such that $M_\phi(1) = \phi(0)1$ and

$$M_\phi(a_1 a_2 \dots a_n) = \phi(n) a_1 a_2 \dots a_n$$

whenever $a_j \in \mathring{\mathcal{A}}_{i_j} = ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \dots \neq i_n$.
Moreover $\|M_\phi\|_{cb} \leq \|\phi\|_{\mathcal{C}}$.

Example

Let $\mathbb{D} = \{s \in \mathbb{C} || s| < 1\}$. For every $s \in \mathbb{D}$

$$\phi_s(n) = s^n$$

defines a radial multiplier M_ϕ on $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ with

$$\|M_{\phi_s}\|_{cb} \leq \frac{|1-s|}{1-|s|}.$$

Strategy

- Uniqueness of M_ϕ
- Reduce to $A_i = B(H_i)$
- Equivalent description of M_ϕ
- Construct $\Phi_{x,y}^i$
- Construct T_1, T_2, T
- Show T is M_ϕ
- Estimate $\|T\|_{cb}$

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- Then $M_\phi|_A : A \to A$ with right behaviour
- $\|M_\phi|_\mathcal{A}\|_{cb} \leq \|M_\phi\|_{cb} \leq \|\phi\|_\mathcal{C}$

Notation

- $$H = \mathbb{C}\Omega \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}.$$

  and denote basis by

  $$\Lambda = \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{\gamma_1 \otimes \cdots \otimes \gamma_n | \gamma_j \in \mathring{\Gamma}_{i_j}, i_1 \neq \cdots \neq i_n\}.$$

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- For $\gamma \in H$, define $L_\gamma \in B(H)$ as

  $$L_\gamma(\chi) = \begin{cases} \gamma \otimes \chi & \text{if } i \neq i_1 \\ 0 & \text{if } i = i_1 \end{cases}$$

  For $\eta, \xi \in H$ let *case 2* if $\eta_{|\eta|}, \xi_{|\xi|} \in H_i$ and *case 1* otherwise.

Equivalent description of M_ϕ

Lemma

Let $T : B(H) \to B(H)$ be a bounded linear normal map, and let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function on $\mathbb{N}_0$. TFAE

Equivalent description of M_ϕ

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(a) $T(1) = \phi(0)1$ *and*

$$T(a_1 a_2 \dots a_n) = \phi(n) a_1 a_2 \dots a_n$$

$$\text{whenever } a_j \in B(\mathring{H}_{i_j}) = \ker(\omega_{i_j}) \text{ and } i_1 \neq i_2 \neq \cdots \neq i_n.$$

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whenever $a_j \in B(\mathring{H}_{i_j}) = ker(\omega_{i_j})$ *and* $i_1 \neq i_2 \neq \cdots \neq i_n$.

(b) *For all* $k, l \in \mathbb{N}_0$ *and* $\xi \in \Lambda(k), \eta \in \Lambda(l)$ *we have*

$$T(L_\xi L_\eta^*) = \begin{cases} \phi(k+l) L_\xi L_\eta^* & \text{in case 1} \\ \phi(k+l-1) L_\xi L_\eta^* & \text{in case 2.} \end{cases}$$

More notation

- For $\gamma \in H$, define $R_\gamma \in B(H)$ as
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- For $a = (a_i) \in l^\infty(\mathbb{N}_0)$ let
  $D_a(\xi_1 \otimes \cdots \otimes \xi_n) = a_n \xi_1 \otimes \cdots \otimes \xi_n$ and $D_a(\Omega) = a_0 \Omega$

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- $\rho(a) = \sum_{\gamma \in \Lambda(1)} R_\gamma a R_\gamma^*$

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- $\rho(a) = \sum_{\gamma \in \Lambda(1)} R_\gamma a R_\gamma^*$
- $\epsilon(a) = \sum_{i \in I} q_i a q_i$ for q_i the projection on

$$\text{span}\{\xi \in \Lambda(n) | n \geq 1, \xi = \gamma_1 \otimes \cdots \otimes \gamma_n, \gamma_n \in \mathring{\Gamma}_i\}$$

Construction of maps

For $x, y \in l^2(\mathbb{N}_0)$ and $a \in B(H)$ put

- $\Phi^{(1)}_{x,y}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D^*_{(S^*)^n y} + \sum_{n=1}^{\infty} D_{S^n x} \rho^n(a) D^*_{S^n y}$

Construction of maps

For $x, y \in l^2(\mathbb{N}_0)$ and $a \in B(H)$ put

- $\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x} \rho^n(a) D_{S^n y}^*$
- $\Phi_{x,y}^{(2)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x} \rho^{n-1}(\epsilon(a)) D_{S^n y}^*$

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- $\Phi^{(2)}_{x,y}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D^*_{(S^*)^n y} + \sum_{n=1}^{\infty} D_{S^n x} \rho^{n-1}(\epsilon(a)) D^*_{S^n y}$

Lemma

For $\xi \in \Lambda(k), \eta \in \Lambda(l)$ we have

$$\rho^n(L_\xi L_\eta^*) = L_\xi L_\eta^* P_{\{\zeta \in H | |\zeta| \leq l+n\}}$$

$$\epsilon(L_\xi L_\eta^*) = \begin{cases} \rho(L_\xi L_\eta^*) & \text{in case 1} \\ L_\xi L_\eta^* & \text{in case 2} \end{cases}$$

Properties of $\Phi^{(i)}_{x,y}$

Lemma

If $\xi \in \Lambda(k), \eta \in \Lambda(l)$ *then*

$$\Phi^{(1)}_{x,y}(L_\xi L_\eta^*) = \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)} \right) L_\xi L_\eta^*$$

and

$$\Phi^{(2)}_{x,y}(L_\xi L_\eta^*) = \begin{cases} \sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)} L_\xi L_\eta^* & \text{case 1} \\ \sum_{t=0}^{\infty} x(k+t-1)\overline{y(l+t-1)} L_\xi L_\eta^* & \text{case 2} \end{cases}$$

Properties of $\phi \in \mathcal{C}$

Lemma

With

$$\phi(n) = \psi_1(n) + \psi_2(n) + c$$

$$\psi_1(k+l) = \sum_{i=1}^{\infty} \sum_{t=0}^{\infty} x_i(k+t)\overline{y_i(l+t)} \tag{1}$$

$$\psi_2(k+l) = \sum_{i=1}^{\infty} \sum_{t=0}^{\infty} z_i(k+t)\overline{w_i(l+t)}.$$

for $h = \sum_{i=1}^{\infty} x_i \odot y_i$ *and* $k = \sum_{i=1}^{\infty} z_i \odot w_i$

Construction of maps 2

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- $T = T_1 + T_2 + cI$

Construction of maps 2

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- $T_1 = \sum_{i=1}^{\infty} \Phi_{x_i,y_i}^{(1)}$ for $h = \sum_{i=1}^{\infty} x_i \odot y_i$
- $T_2 = \sum_{i=1}^{\infty} \Phi_{z_i,w_i}^{(2)}$ for $k = \sum_{i=1}^{\infty} z_i \odot w_i$
- $T = T_1 + T_2 + cI$
- $\cdots$ T has the right behavior

Estimate $\|T\|_{cb}$

- $\|\Phi^{(1)}_{x_i,y_i}\|_{cb} \leq \|x_i\|_2 \|y_i\|_2$

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- $\|\Phi^{(2)}_{z_i,w_i}\|_{cb} \leq \|z_i\|_2 \|w_i\|_2$
- $\|T_1\|_{cb} \leq \sum_{i=1}^{\infty} \|\Phi^{(1)}_{x_i,y_i}\|_{cb} \leq \|h\|_1$

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- $\|T\|_{cb} \leq \|T_1\|_{cb} + \|T_2\|_{cb} + \|cId\|_{cb} \leq \|\phi\|_{\mathcal{C}}$

von Neumann algebra version

Theorem (Haagerup-M 2010)

Let $(\mathcal{M},\omega) = \bar{*}_{i\in I}(\mathcal{M}_i,\omega_i)$ *be the w*-reduced free product of von Neumann algebras* $(\mathcal{M}_i)_{i\in I}$ *with respect to normal states* $(\omega_i)_{i\in I}$ *for which the GNS-representation* π_{ω_i} *is faithful for all* $i \in I$*.*
If $\phi \in \mathcal{C}$*, then there is an unique linear completely bounded normal map*

$$M_\phi : \mathcal{M} \to \mathcal{M}$$

such that $M_\phi(1) = \phi(0)1$ *and*

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Moreover $\|M_\phi\|_{cb} \leq \|\phi\|_{\mathcal{C}}$*.*

Open questions

- For which $(A_i, \omega_i)_{i\in I}$ holds $\|M_\phi\|_{cb} = \|\phi\|_{\mathcal{C}}$ for all $\phi \in \mathcal{C}$?
  - [Wysoczanski 1995] True if $A_i = C_r^*(G_i)$, $|G_i| = \infty$, $|I| = \infty$

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- Use $(\phi_k)_k \subset \mathcal{C}$ with finite support, pointwise converging to 1 on CCAP
- Almagamated counterpart
  - scalar valued ϕ
  - B-valued ϕ