The generator problem for $\mathcal{Z}$-stable C^*-algebras.

Hannes Thiel, University of Copenhagen
(joint work with Wilhelm Winter, University of Münster)

15. September 2011
NordForsk Network Junior Workshop
"Operator Algebra and Dynamics"
University of Copenhagen

The generator problem for vN. algebras

- for $S \subset B(H)$, consider generated C^*-algebra $C^*(S)$ and von Neumann algebra $W^*(S)$
- generator problem: Which algebras arise for $S = \{x\}$ (then called singly generated) ?

Question 1.1 (Kadison 1967 Problem 14)

Is every separably-acting vN. algebra singly generated?

verified for the following classes:

- abelian (von Neumann 1931)
- type I (Pearcy 1962)
- properly infinite (Wogen 1969)

Thus reduced to II_1 case. Further reduced to II_1-factor (Willig 1974). For II_1-factors verified for:

- with Cartan subalgebra (Popa 1985)
- property Γ or tensorially non-prime (Ge, Popa 1998)

Not known if free group factors $W^*(F_k)$ are singly generated.

The generator problem for C^*-algebras I

Definition 1.2

gen(A) := *smallest number* $n \in \{1, 2, 3, \ldots, \infty\}$ *s.t.* A *contains generating subset* $S \subset A_{sa}$ *of cardinality* n.

- If $a, b \in A_{\text{sa}}$, then $\{a, b\}$ generate same sub-C^*-algebra as $a + ib$. Therefore: call A singly generated $:\Leftrightarrow$ gen(A) ≤ 2.
- gen($C(X)$) $\leq n \Leftrightarrow X \subset \mathbb{R}^n$
- A unital, gen(A) $\leq n^2 + 1 \;\Rightarrow\;$ gen($A \otimes M_n$) ≤ 2

Principle: The more non-commutative, the less generators are needed. Cases of 'maximally non-commutativity':

1. A is simple,
2. A contains a sequence of pairwise orthogonal, full elements,
3. A has no finite-dimensional irreducible representations.

In general, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). (2) $\not\Rightarrow$ (1), (3) $\Rightarrow$ (2) unkonwn (connected to Glimm halving problem).

The generator problem for C^*-algebras II

Question 1.3

Is a separable, unital, simple C^*-algebra singly genereated?

Question 1.4

Is a separable, unital C^*-algebra singly generated if it has no finite-dimensional irreducible representations?

Unital, separable C^*-algebra A is singly generated if:

1. A is properly infinite (i.e. $\mathcal{O}_\infty \subset_1 A$) (Kirchberg)
2. A is UHF algebra (Topping 1968)
3. $A \cong B \otimes C$ with C a UHF algebra (Olsen, Zame 1976)
4. A is approximately divisible (Li, Shen 2008) ($\forall\, F \subset A$ finite, $\varepsilon > 0\ \exists\, B \subset_1 A$ finite-dimensional s.t. B has no characters and $\|xb - bx\| \leq \varepsilon \|b\|$ for all $x \in F, b \in B$)

The generator problem for C^*-algebras III

- All cases (except prop. infinite) generalized by $\mathcal{Z}$-stability ($A \cong A \otimes \mathcal{Z}$).
- Jiang-Su algebra $\mathcal{Z} = \varinjlim \mathcal{Z}_{2^\infty,3^\infty}$
- $\mathcal{Z}_{2^\infty,3^\infty} = \{f\colon [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in 1 \otimes M_{3^\infty}, f(1) \in M_{2^\infty} \otimes 1\}$

Facts:

1. $I \lhd A \ \Rightarrow\ \text{gen}(A/I) \leq \text{gen}(A)$
2. $I \lhd A \ \Rightarrow\ \text{gen}(I) \leq \text{gen}(A) + 1$
3. $I \lhd A \ \Rightarrow\ \text{gen}(A) \leq \text{gen}(I) + \text{gen}(A/I)$
4. A unital $\ \Rightarrow\ \text{gen}(A \otimes M_n) \leq \left\lceil \frac{\text{gen}(A)-1}{n^2} + 1 \right\rceil$ (Ichihara, Nagisa)
5. A unital, $\text{gen}(A) \leq n^2 + 1 \ \Rightarrow\ \text{gen}(A \otimes M_n) \leq 2$

New results I

Lemma 1.5

A unital, separable $\Rightarrow$ $\mathrm{gen}(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) \leq 6$.

Proof.

Consider extension:

$$0 \to A \otimes C_0(0,1) \otimes M_{6^\infty} \to A \otimes \mathcal{Z}_{2^\infty,3^\infty} \to A \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0$$

Olsen, Zame: $\mathrm{gen}(A \otimes B) \leq 2$ for $B = M_{2^\infty}, M_{3^\infty}$ or M_{6^∞}.
$\Rightarrow$ for quotient: $\mathrm{gen}(A \otimes (M_{2^\infty} \oplus M_{3^\infty})) \leq 2$; for ideal:

$$\begin{aligned} \mathrm{gen}(A \otimes C_0(0,1) \otimes M_{6^\infty}) &\leq \mathrm{gen}(C_0(0,1)) + \mathrm{gen}(A \otimes M_{6^\infty}) \\ &\leq 2 + 2 \end{aligned}$$

$\Rightarrow$ extension generated by $2 + 4 = 6$ self-adjoint elements. □

New results II

Corollary 1.6

A unital, separable $\Rightarrow$ $\mathrm{gen}(A \otimes \mathcal{Z}) \leq 12$.

Proof.

Have $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, and unital embedding $\mathcal{Z}_{2^\infty,3^\infty} \subset \mathcal{Z}$.
Consider two sub-C^*-algebras $D_1, D_2 \subset A \otimes \mathcal{Z} \otimes \mathcal{Z}$:

$$D_1 := A \otimes \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}$$
$$D_2 := A \otimes \mathcal{Z} \otimes \mathcal{Z}_{2^\infty,3^\infty}$$

By above lemma, $\mathrm{gen}(D_1), \mathrm{gen}(D_2) \leq 6$.
$A \otimes \mathcal{Z} \otimes \mathcal{Z}$ is generated by D_1 and D_2.
$\Rightarrow \mathrm{gen}(A \otimes \mathcal{Z}) = \mathrm{gen}(A \otimes \mathcal{Z} \otimes \mathcal{Z}) \leq 12$. □

with more work, get:

Theorem 1.7 (T, Winter)

A unital, separable $\Rightarrow$ $\mathrm{gen}(A \otimes \mathcal{Z}) \leq 2$.

New results III

Theorem 1.8 (T, Winter)

A, B unital, separable, with:

1. *A contains a sequence of full, positive elements that are pairwise orthogonal,*
2. *B admits a unital embedding of $\mathcal{Z}$.*

Then $\mathrm{gen}(A \otimes_{max} B) \leq 2$. Other tensor product is quotient of $A \otimes_{max} B$, thus also singly generated.

Corollary 1.9

A, B unital, separable, with A simple and $\mathcal{Z} \subset_1 B \;\Rightarrow\; \mathrm{gen}(A \otimes B) \leq 2$.

non-trivial, but true: $\mathcal{Z} \subset_1 C_r^*(F_k)$ for $k \in \{2, 3, \ldots, \infty\}$

Corollary 1.10

A unital, separable, simple $\;\Rightarrow\; \mathrm{gen}(A \otimes C_r^*(F_k)) \leq 2$.
In particular $\mathrm{gen}(C_r^*(F_\infty) \otimes C_r^*(F_\infty)) \leq 2$.