Overview

Multiplier correspondences and applications to crossed products

Overview of the presentation.

- C^*-correspondences and Cuntz-Pimsner algebras
- Multiplier correspondences
- Crossed product C^*-correspondences

Right Hilbert-modules

Let X be a Banach space and A be a C^*-algebra. Suppose we have a right action $X \times A \to X$ of A on X and an A valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ that satisfies

- $\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle \cdot a$
- $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$
- $\langle \xi, \xi \rangle \geq 0$ and $\|\xi\|_X = \sqrt{\|\langle \xi, \xi \rangle\|_A}$.

for all $\xi, \eta \in X$, $a \in A$.

Then we say X is a *right Hilbert A-module*.

Let X, Y be right Hilbert A-modules.

We say a linear operator $T : X \to Y$ is *adjointable* if there exists an operator $T^* : Y \to X$ such that

$$\langle T(\xi), \eta \rangle = \langle \xi, T^*(\eta) \rangle$$

for all $\xi \in X, \eta \in Y$.

We write $\mathcal{L}(X, Y)$ for the collection of all adjointable operators $T : X \to Y$.

$\mathcal{L}(X) := \mathcal{L}(X, X)$ is a C^*-algebra.

For $\xi \in X, \eta \in Y$, define $\theta_{\eta,\xi} : X \to Y$ to be the operator satisfying

$$\theta_{\eta,\xi}(\zeta) = \eta \cdot \langle \xi, \zeta \rangle.$$

for all $\zeta \in X$.

This is an adjointable operator with $(\theta_{\eta,\xi})^* = \theta_{\xi,\eta}$. We call

$$\mathcal{K}(X, Y) = \overline{\text{span}}\{\theta_{\eta,\xi} : \xi \in X, \eta \in Y\}$$

the *compact* operators.

Then $\mathcal{K}(X) := \mathcal{K}(X, X)$ is a closed two-sided ideal in $\mathcal{L}(X)$ and $\mathcal{L}(X) = M(\mathcal{K}(X))$.

Definition

A C^**-correspondence* is a pair (X, A) where X is a Hilbert A-module, equipped with a $*$-homomorphism

$$\phi_X : A \to \mathcal{L}(X).$$

We call ϕ_X the *left action* of A on X and for $a \in A, \xi \in X$ we write $a \cdot \xi$ for $\phi_X(a)(\xi)$.

We say a C^*-correspondence (X, A) is *nondegenerate* if $\phi_X : A \to \mathcal{L}(X) = M(\mathcal{K}(X))$ is nondegenerate.

Example

Let D be a C^*-algebra. Then (D, D) is a C^*-correspondence with

- $\langle a, b \rangle = a^* b$,
- $a \cdot b = ab$,
- $b \cdot a = ba$

for $a, b \in D$.

In this case we have isomorphisms $\mathcal{K}(D) \cong D$ and $\mathcal{L}(D) \cong M(D)$.

Morphisms

A *morphism* $(\psi_X, \psi_A) : (X, A) \to (Y, B)$ is a pair of maps with $\psi_X : X \to Y$ linear and $\psi_A : A \to B$ a C^*-homomorphism satisfying

- $\langle \psi_X(\xi), \psi_X(\eta) \rangle = \psi_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in X$,
- $\psi_X(\phi_X(a)\xi) = \phi_Y(\psi_A(a))\psi_X(\xi)$ for all $\xi \in X$ and $a \in A$.

Given a morphism, there exists a $*$-homomorphism $\psi_X^{(1)} : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfying

$$\psi_X^{(1)}(\theta_{\eta,\xi}) = \theta_{\psi_X(\eta),\psi_X(\xi)}.$$

Definition (Katsura 2003)

Define an ideal J_X of A by

$$J_X := \{a \in A : \phi_X(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker \phi_X\}$$

We say a morphism $(\psi_X, \psi_A) : (X, A) \to (Y, B)$ is *covariant* if it satisfies

- $\psi_A(J_X) \subset J_Y$,
- $\psi_X^{(1)}(\phi_X(a)) = \phi_Y(\psi_A(a))$ for all $a \in J_X$.

A *covariant representation* of (X, A) on a C^*-algebra D is a covariant morphism

$$(\pi_X, \pi_A) : (X, A) \to (D, D).$$

Definition (Katsura 2003)

The Cuntz-Pimsner algebra is $\mathcal{O}_X$ is defined as $\mathcal{O}_X = C^*(k_X(X), k_A(A))$ where (k_X, k_A) is the universal covariant representation of (X, A).

Given a covariant morphism $(\psi_X, \psi_A) : (X, A) \to (Y, B)$, there exists a C^*-homomorphism

$$\mathcal{O}_{\psi_X} : \mathcal{O}_X \to \mathcal{O}_Y$$

such that

$$\begin{array}{ccc} (X,A) & \xrightarrow{(\psi_X,\psi_A)} & (Y,B) \\ \Big\downarrow{\scriptstyle (k_X,k_A)} & & \Big\downarrow{\scriptstyle (k_Y,k_B)} \\ \mathcal{O}_X & \xrightarrow{\mathcal{O}_{\psi_X}} & \mathcal{O}_Y \end{array}$$

commutes.

In this way, $\mathcal{O}$ defines a covariant functor from the category of C^*-correspondences to the category of C^*-algebras.

Multipliers

Let (X, A) be a nondegenerate C^*-correspondence. The multipliers of X are defined as

$$M(X) := \mathcal{L}(A, X)$$

Proposition

$(M(X), M(A))$ is a C^*-correspondence with

- $\langle S, T \rangle = S^* \circ T \in \mathcal{L}(A) = M(A)$,
- $S \cdot m = S \circ m$
- $m \cdot S = \overline{\phi_X}(m) \circ S$.

where $S, T \in \mathcal{L}(A, X), m \in M(A) = \mathcal{L}(A)$ and $\overline{\phi_X}$ is the extension of ϕ_X to $M(A)$.

Relative multipliers

If (X, A) is a nondegenerate C^*-correspondence and $\kappa : C \to M(A)$ is a nondegenerate homomorphism, the C*-multipliers* of X are

$$M_C(X) := \{T \in M(X) : \kappa(C) \cdot T \cup T \cdot \kappa(C) \subset X\}.$$

The C*-strict* topology on $M_C(X)$ is generated by the seminorms

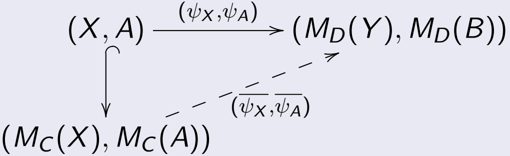
$$m \mapsto \|\kappa(c) \cdot m\| \text{ and } m \mapsto \|m \cdot \kappa(c)\| \text{ for } c \in C.$$

Theorem (Deaconu-Kumjian-Quigg 2011)

Let (X, A) and (Y, B) be nondegenerate C^*-correspondences, let $\kappa : C \to M(A)$ and $\sigma : D \to M(B)$ be nondegenerate homomorphisms and let $(\psi_X, \psi_A) : (X, A) \to (M_D(Y), M_D(B))$ be a morphism. If there is a nondegenerate homomorphism $\lambda : C \to M(\sigma(D))$ such that

$$\psi_A(\kappa(c)a) = \lambda(c)\psi_A(a) \text{ for } c \in C, a \in A,$$

then there is a unique C-strict to D-strictly continuous correspondence homomorphism $(\overline{\psi_X}, \overline{\psi_A})$ making the following diagram commute.



Definition (Kaliszewski-Quigg-R 2011)

Let (X, A) and (Y, B) be nondegenerate C^*-correspondences. We say a homomorphism $(\psi_X, \psi_A) : (X, A) \to (M(Y), M(B))$ is multiplier covariant if

- $\psi_X(X) \subset M_B(Y)$,
- $\psi_A : A \to M(B)$ *is nondegenerate,*
- $\psi_A(J_X) \subset \{m \in M(B) : mB \cup Bm \subset J_Y\}$, *and*
- *the diagram*

$$\begin{CD} J_X @>{\psi_A}>> M(B) \\ @V{\varphi_X}VV @VV{\overline{\varphi_Y}}V \\ \mathcal{K}(X) @>>{\psi_X^{(1)}}> M_B(\mathcal{K}(Y)) \end{CD}$$

commutes.

Theorem (Kaliszewski-Quigg-R)

Let (X, A) and (Y, B) be nondegenerate C^*-correspondences, and let $(\psi_X, \psi_A) : (X, A) \to (M(Y), M(B))$ be a multiplier covariant homomorphism. Then there is a unique homomorphism Ψ_X making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\psi_X, \psi_A)} & (M_B(Y), M(B)) \\ {\scriptstyle (k_X, k_A)}\downarrow & & \downarrow{\scriptstyle (\overline{k_Y}, \overline{k_B})} \\ \mathcal{O}_X & \xrightarrow[\Psi_X]{} & M_B(\mathcal{O}_Y) \end{array}$$

commute. Moreover, Ψ_X is nondegenerate, and is injective if ψ_A is.

Crossed products

Let (X, A) be a nondegenerate C^*-correspondence, G a locally compact topological group.

Let (γ_X, γ_A) be an action of G on (X, A). For $\xi, \eta \in C_c(G, X), f \in C_c(G, A)$ set

$$(\xi \cdot f)(s) = \int_G \xi(t)\gamma_A(t)(f(t^{-1}s))dt$$
$$\langle \xi, \eta \rangle(s) = \int_G \gamma_A(t^{-1})(\langle \xi(t), \eta(st) \rangle)dt.$$

Then $X \rtimes_{\gamma_X} G$ defined to be the completion of $C_c(G, X)$ is a right Hilbert $A \rtimes_{\gamma_A} G$-module.

Left action

The triple $(\mathcal{K}(X), \gamma_X^{(1)}, G)$ defines a C^*-dynamical system, and

$$\mathcal{K}(X \rtimes_{\gamma_X} G) \cong \mathcal{K}(X) \rtimes_{\gamma_X^{(1)}} G$$

Define the left action on $f \otimes a \in C_c(G) \otimes A \cong C_c(G, A)$ by

$$\begin{aligned} \phi_{X \rtimes_{\gamma_X} G}(f \otimes a) &= f \otimes \phi_X(a) \\ &\in M(\mathcal{K}(X) \rtimes_{\gamma_X^{(1)}} G) \\ &= \mathcal{L}(X \rtimes_{\gamma_X} G) \end{aligned}$$

Then $(X \rtimes_{\gamma_X} G, A \rtimes_{\gamma_A} G)$ is a C^*-correspondence

Covariant maps

We have the canonical embeddings

$$i_G : G \to M(A \rtimes_{\gamma_G} G)$$

and

$$i_A : A \to M(A \rtimes_{\gamma_A} G).$$

We can define a map

$$i_X : X \to M_{A \rtimes_{\gamma_A} G}(X \rtimes_{\gamma_X} G)$$
$$(i_X(\xi)f)(s) = x \cdot f(s)$$

for $\xi \in X, f \in C_c(G, A), s \in G$.

Multiplier covariant morphism

The pair

$$(i_X, i_A) : (X, A) \to (M_{A \rtimes_{\gamma_A} G}(X \rtimes_{\gamma_X} G), M(A \rtimes_{\gamma_A} G))$$

defines a multiplier covariant morphism.

Applying the Theorem we get

$$I_X : \mathcal{O}_X \to M(\mathcal{O}_{X \rtimes_{\gamma_X} G}).$$

If we define $u := \overline{k_{A \rtimes_{\gamma_A} G}} \circ i_G : G \to M(\mathcal{O}_{X \rtimes_{\gamma_X} G})$ then (I_X, u) is a covariant homomorphism of $(\mathcal{O}_X, \mathcal{O}_{\gamma_X}, G)$ in $\mathcal{O}_{X \rtimes_{\gamma_X} G}$.

The integrated form

$$I_X \times u : \mathcal{O}_X \rtimes_{\mathcal{O}_{\gamma_X}} G \to \mathcal{O}_{X \rtimes_{\gamma_X} G}$$

is surjective.

Theorem (Hao-Ng 2008, Kaliszewski-Quigg-R 2011)

When G is amenable there is an isomorphism

$$\mathcal{O}_X \rtimes_{\mathcal{O}_{\gamma_X}} G \cong \mathcal{O}_{X \rtimes_{\gamma_X} G}.$$

THANKYOU