Problems in the representation theory and cohomology of groups

Contents

1. Ordinary character theory

Problem 1.1 (Isaacs). Can a noncyclic solvable group G have two faithful irreducible characters whose product is irreducible?

Remarks: If so, then there exists a minimal normal subgroup M which is not central. For any such M, we have $|M| = p^e$ with $p|e$, $p|\alpha(1)$ and $p|\beta(1)$. Also, the fitting height of G is at least 4.

If Conjecture 3.5 is true, then there cannot exist an example. This is proved by taking S to be the character group of M and k to be the field of order p. If λ, μ are irreducible constituents of the restrictions of α and β to M, then H and K are taken to be the stabilizers in G of λ and μ.

Problem 1.2 (Dade). Is it true that if G is an odd order M-group (i.e., every ordinary representation is monomial) and $N \trianglelefteq G$ then N is an M-group?

Remarks: Loukaki has proved in her Ph. D. thesis that the answer is yes if $|G|$ is only divisible by two primes.

2. Modular character theory

Conjecture 2.1 (Alperin's conjecture). Let G be a finite group and k be a field of characteristic p. Then the number of simple kG-modules is equal to the number of ordered pairs consisting of a p-subgroup P of G and a projective simple $kN_G(P)/P$-module.

Remarks: There are various refinements of this conjecture. The easiest one is the blockwise version, where one counts simple modules in a block and projective simples in Brauer corresponding blocks.

E. C. Dade also has various generalizations which have the advantage that they survive Clifford theory better, so that it suffices to prove them for groups which are almost simple in a suitable sense.

Relevant publications: Alperin [4], Dade [31, 32], Knörr and Robinson [47], Külshammer [50], Robinson [69, 71], Robinson and Staszewski [73], Thévenaz [76, 77].

Conjecture 2.2 (Alperin–McKay conjecture). Let $k_0(B)$ denote the number of irreducible ordinary characters of height zero in a block B of the group algebra of a finite group. Let D be a defect group of B, and let b be the block of $N_G(D)$ Brauer correspondent to B. Then $k_0(B) = k_0(b)$.

Relevant publications: Alperin [3], Külshammer [50], Külshammer and Robinson [52], Lehrer [53], Michler and Olsson [55].

Conjecture 2.3 (Brauer's $k(B)$ conjecture). Let $k(B)$ be the number of ordinary irreducible characters in a block B of the group algebra of a finite group, and let D be a defect group of B. Then $k(B) \leq |D|$.

Remarks: Brauer and Feit have shown that $k(B) \leq (|D|^2/4) + 1$.

Relevant publications: Brauer [18] (Problem 20), Brauer and Feit [19], Knörr [46], Nagao [59], Robinson [68, 72], Robinson and Thompson [74].

Conjecture 2.4 (Brauer's Problem 21). There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if B is a block of a group algebra of a finite group, with defect group D, then $|D| \leq f(k(B))$.

Remarks: Külshammer and Robinson have shown, using Zelmanov's solution of the restricted Burnside problem, that the Alperin–McKay Conjecture 2.2 implies an affirmative answer to Brauer's Problem 21.

Relevant publications: Brauer [18] (Problem 21), Külshammer and Robinson [52],

Conjecture 2.5 (Brauer's Cartan invariant conjecture). There is a function of the order of the defect groups of a block B of a finite group algebra which gives an upper bound for the Cartan invariants c_{ij} of the block B.

Relevant publications: Brauer [18] (Problem 22). Originally stated with the identity function. I think Landrock found a counterexample.

Conjecture 2.6 (R. Brauer's height zero conjecture). Every irreducible ordinary character in a block B of the group algebra of a finite group has height zero if and only if B has abelian defect groups.

Relevant publications: Brauer [18] (Problem 23), Murai [58].

Conjecture 2.7 (Nagao's $k(GV)$ conjecture). Let G be a p'-group acting faithfully on a $\mathbb{F}_pG$-module V. Then the semidirect product $V \rtimes G$ has at most $|V|$ conjugacy classes.

Remarks: Nagao's $k(GV)$ Conjecture 2.7 implies the p-solvable case of Brauer's $k(B)$ Conjecture 2.3.

Relevant publications: Gow [39], Knörr [45, 46], Nagao [59], Robinson [70], Robinson and Thompson [74].

Conjecture 2.8 (Olsson's conjecture). Let B be a block of a group algebra with defect group D, and let $k_0(B)$ be defined as in Conjecture 2.2. Then $k_0(B) \leq |D/[D,D]|$.

Remarks: Külshammer has shown that the Alperin–McKay Conjecture 2.2 and Nagao's $k(GV)$ Conjecture 2.3 together imply Olsson's Conjecture 2.8.

Relevant publications: Külshammer [49], Olsson [62].

Conjecture 2.9 (Harada's conjecture)**.** Let S be a subset of the set of ordinary irreducible characters of a finite group G, and let p be a prime number. If the ordinary character $\sum_{\chi\in S}\chi(1).\chi$ vanishes on p-singular elements, then S is a union of p-blocks of G.

Relevant publications: Harada [40].

Problem 2.10. Find the decomposition numbers for the finite symmetric groups.

Remarks: A lot of partial information is known on this problem.

Relevant publications: James [44].

Conjecture 2.11 (Gow–Kleshchev)**.** The only example of faithful simple modules M and N for a symmetric group Σ_n over a field k such that $M \otimes N$ is simple is the case

$$D^{(2l+2,2l)} \otimes D^{(4l-2j+1,2j+1)} \cong D^{(2l+1-j,2l-j,j+1,j)}$$

where $n = 4l+2$ and $0 \leq j < l$, and where the characteristic of k is two.

Remarks: Gordan James has proved that the stated tensor product is correct.

Problem 2.12. Which complex representations of Σ_n are irreducible modulo p?

Remarks: The answer is known if $p = 2$ by work of James and Mathas, and for representations indexed by p-regular partitions, by work of James, and of James and Murphy.

3. Modular representation theory

Conjecture 3.1 (Broué's conjecture)**.** Let G be a finite group and let k be a field of characteristic p. Let B be a block of kG with abelian defect group D, and let b be the Brauer corresponding block of $kN_G(D)$. Then there is an equivalence of derived categories between $D^b(\mathsf{Mod}(B))$ and $D^b(\mathsf{Mod}(b))$.

Remarks: It is known that the same is not true for nonabelian T.I. defect groups, for example it fails for the principal blocks of $Sz(8)$ and its Sylow 2-normalizer.

Relevant publications: Broué [21, 22, 23], Külshammer [50].

Conjecture 3.2 (Donovan's conjecture)**.** Given a field k of characteristic p and a p-group D, there are only a finite number of Morita types of blocks of group algebras of finite groups over k with defect group D.

Remarks: Donovan's conjecture implies an affirmative answer to Brauer's Cartan invariant conjecture 2.5.

Relevant publications: Külshammer [51].

Conjecture 3.3 (Feit's conjecture)**.** Let P be a finite p-group and k be a field of characteristic p. Then there are only a finite number of finite dimensional kP-modules which are sources for simple kG-modules, as G runs over the finite groups containing P as a subgroup.

Conjecture 3.4 (Puig's conjecture)**.** Let P be a finite p-group and let k be a field of characteristic p. Then there are only finitely many isomorphism classes of finite dimensional interior P-algebras over k which are source algebras of blocks of kG, as G runs over the finite groups containing P as a subgroup.

Remarks: Puig's conjecture implies Donovan's conjecture 3.2.

Conjecture 3.5 (Isaacs)**.** Let k be a field and G be a finite group. Let S be a simple kG-module. Assume that G factorizes as HK, where H and K are two subgroups which both have nontrivial fixed points on S. Then the module S is trivial.

Remarks: If there is a counterexample, then k has prime characteristic p which divides $|H|$ and $|K|$. If G is solvable then also p divides $|G : H|$, $|G : K|$ and $\dim S$.

Another way of expressing the conjecture is as follows. There cannot exist permutation modules M and N of G over k such that $\dim_k \mathrm{Hom}_{kG}(M, N) = 1$, and such that the socles of M and N have isomorphic nontrivial simple submodules. This makes it clear why k has to have characteristic p.

Problem 3.6. Classify the finite dimensional indecomposable representations of the generalized quaternion groups in characteristic two.

Remarks: The representation type of a finite group G over a field of characteristic p only depends on the Sylow p-subgroup. The representation type is finite if G has cyclic Sylow p-subgroups, tame if $p = 2$ and the Sylow 2-subgroups are dihedral, semidihedral or generalized quaternion. The finite dimensional indecomposable representations for the dihedral groups were classified by Ringel, and those for the semidihedral groups by Crawley–Boevey.

Relevant publications: Ringel [67].

Problem 3.7. The finitely generated indecomposable modules for the Klein four group $\mathbb{Z}/2 \times \mathbb{Z}/2$ have been determined both over the integers and over a field of characteristic two. What does reduction modulo two do?

Relevant publications: Ringel [67].

Problem 3.8. Do there exist nilpotent elements in the representation ring of an elementary abelian group of order 2^n, $n \geq 3$ over a field of characteristic two?

Remarks: It is known that in characteristic p odd, there are nilpotent elements if and only if the Sylow p-subgroups are cyclic. In characteristic two, it is known that if the Sylow p-subgroups are not cyclic or elementary abelian, there are nilpotent elements. The only cases open are the elementary abelian Sylow 2-subgroups of order 2^n, $n \geq 3$.

Relevant publications: Benson and Carlson [10], Zemanek [79, 80].

Problem 3.9. The finitely generated indecomposable modules for the dihedral groups in characteristic two have been classified by Ringel. How do the tensor products decompose? Find the structure of the representation ring, and understand its nilpotent elements. Are there any nilpotent elements which do not cube to zero?

Relevant publications: Benson and Carlson [10], Ringel [67], Zemanek [79, 80].

Problem 3.10 (Carlson)**.** Let M be a finitely generated kG-module in a thick subcategory $\mathcal{M}$ of the stable category of finitely generated kG-modules $\mathsf{stmod}(kG)$. If $\mathrm{Ext}^*_{kG}(M, N) \neq 0$ for all nonprojective modules N in $\mathcal{M}$, does it follow that M generates $\mathcal{M}$?

Problem 3.11. What is the pure global dimension of kG, where k is a field of characteristic two and G is a dihedral group of order at least eight, a semidihedral group, or a generalized quaternion group? The pure global dimension is known in all other cases.

Relevant publications: Benson and Gnacadja [16].

Problem 3.12. Let k be a field and G a finite group. Gnacadja defines a map of kG-modules $f : M \to N$ to be phantom if the restriction of f to every finitely generated submodule of M factors through a projective modules.

Is there a bound depending only on G and not on k, for the number of phantom maps between kG-modules we need to compose, to obtain a map which factors through a projective module? The answer is yes in characteristic two for groups of 2-rank two.

Remarks: If k has cardinality $\aleph_n$ for $n < \omega$ then it is known that the composite of $n+2$ phantom maps factors through a projective module. An example is known of two phantom maps whose composite does not factor through a projective module.

Relevant publications: Benson [8], Benson and Gnacadja [16], Gnacadja [38].

4. Modular invariant theory

Problem 4.1. Let G be a finite group acting linearly on a vector space V over a field k. Is it true that the ideal of $k[V]$ generated by the invariants of positive degree is generated in degrees at most $|G|$?

Remarks: The answer is yes if the characteristic of k does not divide $|G|$. If the characteristic does divide $|G|$, examples are known where the ring $k[V]^G$ is not generated in degrees at most $|G|$.

5. Rings and modules

Conjecture 5.1 (M. Auslander)**.** Let Λ and Γ be Artin algebras which are stably equivalent. Then the number of nonprojective simple Λ-modules is the same as the number of nonprojective simple Γ-modules.

Conjecture 5.2 (T. Nakayama)**.** Let Λ be an Artin algebra. If the modules in the minimal injective resolution of Λ as a left Λ-module are projective then Λ is self-injective.

Relevant publications: Fuller and Zimmermann-Huisgen [37], Wilson [78].

Conjecture 5.3 (Finitistic dimension conjecture)**.** Let Λ be an Artin algebra. Then the finitistic projective dimension of Λ is finite. In other words, there is an upper bound for the projective dimensions of Λ-modules of finite projective dimension.

Remarks: A weaker conjecture would be that there is an upper bound for the projective dimensions of finitely generated Λ-modules of finite representation type.

Problem 5.4. For Artin algebras, is wild representation type equivalent to undecidability of the theory of modules?

Remarks: The answer is yes for finite dimensional hereditary algebras.

Relevant publications: Prest [64].

Problem 5.5. Let Λ and Γ be Artin algebras. Suppose that the stable module categories $\underline{\mathsf{mod}}(\Lambda)$ and $\underline{\mathsf{mod}}(\Gamma)$ of finitely generated modules are equivalent. Does it follow that the stable module categories $\underline{\mathsf{Mod}}(\Lambda)$ and $\underline{\mathsf{Mod}}(\Gamma)$ of all modules are equivalent?

Problem 5.6. Let K be a field. Two finite dimensional K-algebras A and B are *stably equivalent of Morita type* if there exist ${}_AN_B$, ${}_BM_A$ such that (i) each is projective as an A-module and as a B-module, (ii) ${}_BM_A \otimes_A {}_AN_B \cong {}_BB_B \oplus$ projective B-B-bimodule, (iii) ${}_AN_B \otimes_B {}_BM_A \cong {}_AA_A \oplus$ projective A-A-bimodule.

If A and B are stably equivalent of Morita type, and so are C and D, does it follow that $A \otimes_K C$ and $B \otimes_K D$ are?

It is not even known whether, if A and B are stably equivalent of Morita type, then so are $A \otimes_K A^{\mathrm{op}}$ and $B \otimes_K B^{\mathrm{op}}$.

Problem 5.7. Over an algebraically closed field, if two finite dimensional algebras (with no semisimple direct factors) are stably equivalent, is there necessarily a stable equivalence of Morita type?

Remarks: If k is not algebraically closed, examples are known, where one of the algebras is self-injective and the other isn't.

Problem 5.8. If Λ is an Artin algebra whose Auslander–Reiten quiver is connected, is it true that Λ has to be of finite representation type?

Conjecture 5.9 (No loops conjecture, Ed Green)**.** Let Λ be an Artin algebra and S a simple Λ-module. Suppose that $\mathrm{Ext}^1_\Lambda(S,S) \neq 0$. Then Λ has infinite global dimension.

Remarks: Igusa has proved the no loops conjecture for finite dimensional algebras over an algebraically closed field, using cyclic homology. The conjecture remains open in general, even for finite dimensional algebras over a field.

Conjecture 5.10. A stronger form of the no loops conjecture states that with the same hypotheses, S has infinite projective dimension.

Remarks: This form of the conjecture is not even known for finite dimensional algebras over an algebraically closed field.

Conjecture 5.11. An even stronger form of the no loops conjecture is that with the same hypotheses, for all $n \geq 0$ there exists $m \geq n$ such that $\mathrm{Ext}^m_\Lambda(S,S) \neq 0$.

Problem 5.12 (de la Peña)**.** Is it true that a tree algebra Λ is derived tame if and only if the Euler form

$$\chi_\Lambda = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{Ext}^i_\Lambda(-,-)$$

is positive semidefinite?

Remarks: A *tree algebra* is defined to be a finite dimensional algebra which is a quotient of the path algebra of a tree over a field k, by an ideal contained in the paths of length at least two. Tree algebras always have finite global dimension, so that the Euler form on the Grothendieck group (with respect to all exact sequences) is equivalent to the form given by $\dim_k \mathrm{Hom}_\Lambda(-,-)$ on $K_0(\Lambda)$. Derived tame means that the derived category of bounded complexes of finitely generated modules is tame. This is not equivalent to tameness of the module category.

Problem 5.13. Let $R \subseteq S$ be rings, with S finitely generated as an R-module. Can there exist a flat S-module which is projective over R but not over S?

Conjecture 5.14. If P and Q are finite p-groups such that the modular group rings $\mathbb{F}_pP$ and $\mathbb{F}_pQ$ are isomorphic then P and Q are isomorphic.

Remarks: A preprint by Borge and Laudal is circulating, which purports to prove this conjecture. It appears that this preprint contains a significant gap.

M. Hertweck has given examples of nonisomorphic finite solvable groups G and H such that the integral group rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic as augmented algebras.

Relevant publications: Drensky [34], Hertweck [43].

6. Finite group actions

Conjecture 6.1. Let G be a finite group, and let r be the maximal rank of an elementary abelian subgroup of G for any prime. Then the following hold.

(i) If G acts freely on a finite CW complex with the homotopy type of a product of s spheres (possibly of different dimensions), then $s \geq r$.

(ii) There exists a free action of G on a finite CW complex with the homotopy type of a product of r spheres (possibly of different dimensions), and with trivial action on homology.

Remarks: In the case where $r = 1$ the conjecture was proved by Swan. The case $r \leq 3$ of (i) has been proved by Heller. In the case $G = S_3$, the symmetric group on three letters, it follows from a theorem of Milnor that there is no free action on an actual sphere of any dimension. Oliver has shown that in the case $G = A_4$, there is no free action on any finite CW complex with the homotopy type of a product of spheres of the same dimension. But there is a free action of A_4 on $S^2 \times S^3$.

Relevant publications: Benson and Carlson [11], Browder [24], Carlsson [28], Conner [29], Heller [41, 42], Lewis [54], Milnor [56], Oliver [61], Swan [75].

Problem 6.2 (A. Adem and E. Yalçin)**.** If $(\mathbb{Z}/2)^r$ acts freely on a finite complex X with mod 2 cohomology generated by one dimensional classes, does it follow that $r \leq 2\dim H^1(X;\mathbb{F}_2)$?

Remarks: Adem and Yalçin showed that an affirmative answer to this question implies that there is a finite 2-group G with rank r providing a counterexample to part (ii) of Conjecture 6.1.

Relevant publications: Adem and Yalçin [2].

Conjecture 6.3 (G. Carlsson)**.** Let k be a field of characteristic p and let G be an elementary abelian p-group of rank r. Then a nonzero finite complex of projective kG-modules

$$0 \to C_t \to \cdots \to C_0 \to 0$$

always satisfies

$$\sum_{i=0}^{t} \dim H_i(C_*) \geq 2^r.$$

Remarks: This conjecture implies part (i) of Conjecture 6.1.

Conjecture 6.4 (W. Dicks). A finite group acting admissibly on a 2-dimensional contractible CW complex has a fixed point.

Conjecture 6.5. If G is a finite complex reflection group acting on $\mathbb{C}^n$, and X denotes $\mathbb{C}^n$ with the reflecting hyperplanes removed, then X is a $K(\pi, 1)$. In most cases this is known to be true, but a uniform proof would be desirable.

Remarks: Brieskorn conjectured this in the case of Coxeter groups, for which there is a uniform proof by Deligne using the theory of buildings. Saito made a more general conjecture: that for a free complex hyperplane arrangement, the complement is a $K(\pi, 1)$. The reflecting hyperplanes for a complex reflection group are examples of free arrangements. A counterexample to Saito's conjecture was found by Edelman and Reiner.

Relevant publications: Brieskorn [20], Deligne [33], Edelman and Reiner [35], Orlik and Terao [63] (section 1.1 defines free hyperplane arrangements, chapter 5 discusses the topology of the complement, and section 6.6 discusses Deligne's proof).

7. Cohomology of finite groups

Problem 7.1. Let k be a field of characteristic p and G be a finite group. If S is a simple module in the principal block of kG, is it true that $H^n(G, S) \neq 0$ for some $n \geq 0$?

There is also an integral version of this problem. If R is a completion at a prime ideal of the ring of integers in an algebraic number field, and M is an irreducible RG-lattice, is it true that $H^n(G, M) \neq 0$ for some $n \geq 0$?

Relevant publications: Benson, Carlson and Robinson [15].

Problem 7.2. Let k be a field of characteristic p, and G a finite group. If $H^1(G, k) \neq 0$ (i.e., if G has a normal subgroup of index p), is it true that for all $n \geq 0$ we have $H^{2n}(G, k) \neq 0$?

Problem 7.3. Let k be a field of characteristic p, and let G be a finite simple group. If S and S' are simple kG-modules, how big can $\mathrm{Ext}^1_{kG}(S, S')$ be? Is there a universal upper bound for the dimension (not depending on k and G)?

Remarks: Of course, if G is a p-group, $\mathrm{Ext}^1_{kG}(k, k)$ can be arbitrarily large. But the hypothesis that G is simple seems too restrictive. What is the right condition? Maybe faithful simple.

Problem 7.4. Does there exist an integer $n > 0$ such that if G is a finite group with $H_i(G, \mathbb{Z}) = 0$ for $i \leq n$ then G is trivial?

Remarks: Adem and Milgram have shown that for G equal to the Mathieu group M_{23}, $H_i(G, \mathbb{Z}) = 0$ for $i \leq 4$. So if such an n exists, it is at least five.

Problem 7.5. Find a group theoretic characterization of those p-groups G with the property that the sum of the restriction maps from $H^*(G, \mathbb{F}_p)$ to the sum of the cohomology rings of all proper subgroups is injective.

Relevant publications: Adem and Karagueuzian [1].

Problem 7.6. Define a sequence $\zeta_1, \dots, \zeta_r$ in $H^*(G,k)$ to be a quasi-regular sequence if ζ_1 is a regular element (i.e., multiplication by ζ_1 is injective on $H^*(G,k)$), and for $i = 2, \dots, r$, multiplication by ζ_i gives an injective map on $H^*(G,k)/(\zeta_1, \dots, \zeta_{i-1})$ in degrees greater than or equal to $\deg(\zeta_1) + \dots + \deg(\zeta_{i-1})$.

For a finite group G, does there always exist a quasi-regular sequence of length equal to the p-rank of G (i.e., equal to the Krull dimension of $H^*(G,k)$)?

Relevant publications: Benson [9], Benson and Carlson [12], Carlson [27, 26].

Conjecture 7.7. A strengthened version of problem 7.6 is the conjecture that the Castelnuovo–Mumford regularity $\mathsf{Reg}\, H^*(G,k)$ is equal to zero.

Remarks: For a generalization of this conjecture, see Conjecture 9.2.

Relevant publications: Benson [7, 9].

Problem 7.8 (Carlson)**.** Let G be a finite group and k be a field. Does $H^*(G,k)$ always have an associated prime $\mathfrak{p}$ such that the Krull dimension of $H^*(G,k)/\mathfrak{p}$ is equal to the depth of $H^*(G,k)$?

Remarks: Let $k[c_1, \dots, c_r]$ be a copy of the Dickson invariants over which $H^*(G,k)$ is finitely generated as a module. A theorem of Bourguiba and Zarati states that if the depth of $H^*(G,k)$ is d then $c_1, \dots, c_d$ is a regular sequence. It may be true that there is an associated prime lying over $(c_{d+1}, \dots, c_r)$. This is not true in general for unstable algebras of finite type over the Dickson invariants.

Relevant publications: Bourguiba and Zarati [17], Carlson [26].

Problem 7.9. Find a small periodic projective resolution over $\mathbb{Z}$ for the groups

$$(\mathbb{Z}/a \times \mathbb{Z}/b \times \mathbb{Z}/c) \rtimes Q_{2^n},$$

where a, b and c are positive integers which are pairwise coprime. Here, Q_{2^n} denotes the generalized quaternion group of order 2^n, acting on the three factors through the three quotients of order two via negation. For what values of a, b and c is there a free resolution of period four? An important special case is the groups $(\mathbb{Z}/p \times \mathbb{Z}/q) \rtimes Q_8$ where p and q are distinct primes. In this case, one can start with the deficiency zero presentation

$$\langle x, y \mid x^p y = y^{-1} x^p,\ xy = y^{2q-1} x^{-1} \rangle$$

(Neumann [60]). Do the remaining cases also have deficiency zero presentations?

Problem 7.10. Develop the theory of "cohomology varieties" for infinitely generated kG-modules where G is a finite group and k is a field which is not algebraically closed, or perhaps a more general commutative ring. The "variety" in this generality should presumably be a subset of the projective scheme $\mathsf{Proj}\, H^*(G,k)$.

Is there a version of Dade's lemma over commutative rings of coefficients which are not fields?

Relevant publications: Benson, Carlson and Rickard [13, 14], Dade [30].

Problem 7.11. A theorem of Mislin states that a map of finite groups induces an isomorphism in mod p cohomology if and only if it induces an equivalence of categories of nontrivial p-subgroups, where the morphisms are generated by conjugations and inclusions. His proof makes essential use of the fact that Haynes Miller has proved Sullivan's homotopy fixed point conjecture. Is there an algebraic proof which makes no use of this theorem?

Relevant publications: Mislin [57].

Problem 7.12. Under what conditions does a map of finite groups induce an inseparable isogeny (i.e., a bijection at the level of underlying sets) between the mod p cohomology varieties? If p is odd, is it true that this condition together with the condition that the Sylow p-subgroups of the two groups have the same order is equivalent to the condition that the map induces an isomorphism in mod p cohomology?

Relevant publications: Quillen [65, 66].

Problem 7.13. Given a split extension $G = N \rtimes H$ and a field k, it is not always true that $E_2 = E_\infty = H^*(G,k)$ for the Lyndon–Hochschild–Serre spectral sequence $H^*(H, H^*(N,k)) \Rightarrow H^*(G,k)$. But this is true for a wreath product of finite groups, by a theorem of Nakaoka. Find other conditions under which this holds. For example, is it true if k has characteristic p, and N is a finite p-group that has a central series for which each layer forms a projective $\mathbb{F}_pH$-module?

As a nontrivial example, the Sylow 2-subgroup of M_{22} has a normal subgroup of index two isomorphic to the Sylow 2-subgroup of $L_3(4)$, and the extension splits. The above condition and conclusion hold in this case.

8. Infinite discrete groups

Conjecture 8.1 (Zero divisor conjecture)**.** If G is a torsion free group and k is a field then the group algebra kG has no nontrivial zero divisors.

Relevant publications: Farkas and Linnell [36].

Problem 8.2 (Leary)**.** Let G be a torsion-free group and k be a field. Can $\tilde{K}_0(kG)$ ever be nontrivial? One might wish to generalize this to let k be an integral domain satisfying $\tilde{K}_0(k) = 0$.

Problem 8.3. Calculate $\tilde{K}_0(\mathbb{Z}F)$, where F is Richard Thompson's group

$$F = \langle x_0, x_1, \dots \mid x_n^{x_i} = x_{n+1}\ \forall i < n\rangle = \langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2]\rangle.$$

Relevant publications: Brown and Geoghegan [25].

Problem 8.4. Is the group of Richard Thompson described in Problem 8.3 amenable?

Remarks: If not (this is the expected answer), it provides the first example of a finitely presented group which is not amenable but does not contain a nonabelian free subgroup. An example of a finitely generated group with these properties was first found by Grigorchuk.

Problem 8.5. Does there exist a cardinality $\aleph$ such that if A is a free abelian group on a set of size $\aleph$ and A acts on a finite dimensional contractible complex then there exists a stabilizer of size $\aleph$?

Remarks: It is known by work of Dicks and Kropholler that the smallest possibility for such an $\aleph$ is $\aleph_{\omega+1}$.

Problem 8.6. Let Γ be a group in Kropholler's class $\mathbf{LH}\mathfrak{F}$, let k be a field of characteristic p, and let M be a $k\Gamma$-module of type FP_∞. Let $\alpha_n(M)$ denote the minimal number of generators possible for the nth kernel in a projective resolution of M, and let $p_M(t)$ be the Poincaré series $\sum_{n=0}^{\infty} \alpha_n(M)t^n$. Is $p_M(t)$ equal to the power series expansion of a rational function of t whose poles are at roots of unity?

Relevant publications: Benson [5, 6], Kropholler [48].

9. Compact Lie groups

Problem 9.1. Calculate the cohomology of the classifying space of the compact Lie group $PU(n)$, namely the quotient of the unitary group $U(n)$ by its center of order n.

Conjecture 9.2. Let G be a compact Lie group of dimension d and let k be a field. Write ε for the sign representation of G on k corresponding to the orientation of the conjugation action on the Lie algebra (this is often the trivial representation). Then the Castelnuovo–Mumford regularity of the $H^*(BG;k)$-module $H^*(BG;\varepsilon)$ is equal to $-d$.

Remarks: This is a generalization of Conjecture 7.7.

Relevant publications: Benson [7, 9].

References for the problems

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Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: bensondj@math.uga.edu