Surgery theory and group actions II

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<u>Ex:</u> (G. Bredon) $\exists$ smooth actions of $\mathbb{Z}/n$ on $\mathbb{R}^k$ $k \geq 8$ with no global fixed pt. (e.g. $n=6$, $n=pq$, $p \neq q$)

<u>Thm (Swan, 1960)</u> π periodic finite gp, period $d(\pi)$, then $\exists$ finite dim CW-cpx X with $\pi_1(X,x)=\pi$, $X \simeq S^{n-1}$ for any n multiple of $d(\pi)$.
Homotopy type of X determined by $g \in H^n(\pi;\mathbb{Z}) \cong \mathbb{Z}/|\pi|$
$g \in (\mathbb{Z}/|\pi|)^*$

$$
\begin{array}{ccccccccc}
g & \longleftrightarrow & 0 \to \mathbb{Z} \to P_{n-1} \to & \cdots & \to P_0 \to \mathbb{Z} \to 0 \\
 & & \quad r \downarrow & & \quad \| \, id \\
 & & 0 \to \mathbb{Z} \to P'_{n-1} \to & \cdots & \to P'_0 \to \mathbb{Z} \to 0
\end{array}
$$

$$\theta(rg) = \theta(g) + [\langle r, \Sigma_\pi \rangle] \in \tilde{K}_0(\mathbb{Z}\pi)$$

ideal in $\mathbb{Z}\pi$.

Milnor $2p$ conditions for top free actions on S^{n-1}
i.e. no dihedral subgps.

<u>Read:</u>
<u>Milnor:</u> Killing the ...
Kervaire–Milnor.

②

How to recognize a manifold?

$M^n \subseteq \mathbb{R}^{n+k}$ $k \gg 0$ ν_M stable normal bundle.

- Poincare duality
- stable normal bundle.

Def: X (finite) oriented Poincaré duality space of dimension n.

1) X is ~~n-dimensional~~ finite dimensional ~~(finite)~~ CW-cpx

2) $\exists\ [X] \in H_n(X;\mathbb{Z})$ "fundamental class"

s.t.

$$[X] \cap - : H^i(X;B) \to H_{n-i}(X;B)$$

$H_n^{lf}(\tilde{X};\mathbb{Z})$ B is a $\Lambda = \mathbb{Z}\pi$ - module

Spivak normal fiber space of X

Def: Embed $X \subset \mathbb{R}^{n+k}$

$$\begin{array}{ccc} X & \subset & \mathbb{R}^{n+k} \\ \cap & \cup & \\ & N & = \text{regular neighborhood} \end{array}$$

$\partial N \to N \simeq X$

spherical fibration

ν_X $(k-1)$-spherical fib.

$\mathcal{S}(X) = \{(M^n, f) \mid f: M \xrightarrow{\simeq} X\} / M_1 \xrightarrow[\simeq]{f_1} X$, $W^{n+1} \supset M_1$, $M_2 \xrightarrow{f_2} X$

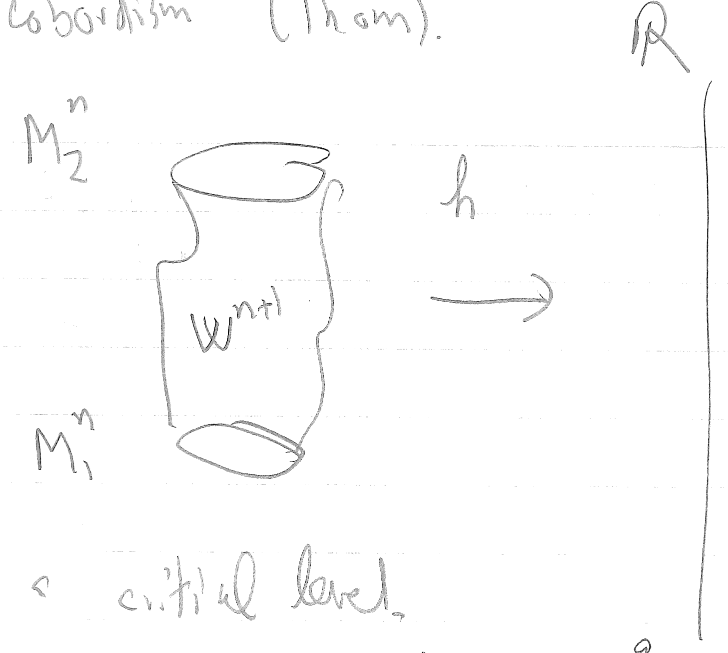
structure set

h-cobordism.

1) $\mathcal{S}(X) \neq \emptyset$

2) Compute it.

Smooth cobordism (Thom).



Change through a critical level.

$$\text{locally} \quad h(x) = h(x^0) + \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j$$

Elementary surgery $\quad \varphi : S^p \times D^{n-p} \hookrightarrow M^n$.

$$\rightsquigarrow \quad M' = \left(M \setminus \varphi(S^p \times D^{n-p})\right) \cup D^{p+1} \times S^{n-p-1}.$$

Surgery exact sequence

$$L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(X^n) \to \mathcal{N}(X) \to L_n(\mathbb{Z}\pi).$$

$$\mathcal{N}(X) = \left\{ \begin{tikzcd} \nu_M \arrow[r, "b"] \arrow[d] & \xi \arrow[d] \\ M^n \arrow[r, "f"] & X \end{tikzcd} \right\} \quad \text{bundle map}$$

$$\left\{ \begin{array}{l} \xi \\ \downarrow \\ X \end{array} \right. \quad \text{vector bundle s.t.} \quad \begin{tikzcd} S(\xi) \arrow[r, "\simeq"] \arrow[dr] & X \arrow[d] \\ & X \end{tikzcd}$$

Example:

$$X_{gen} = \begin{pmatrix} S^2 \\ \downarrow \\ E \\ \downarrow \\ S^3 \end{pmatrix}$$

given by generator in $\pi_2(h\,Aut(S^2))$

$$\| $$

$$\pi_4(S^2)$$

If $\eta(X) \neq \phi$ $\quad \eta(X) = [X, G/O]$, ← smooth

$[X, G/Top]$ ← top.

$\nu_X \notin S(\xi)$.

Poincaré duality case

$$X \xrightarrow{c} K = K(\pi,1)$$

$$L_n(K) = \left\{ \begin{pmatrix} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow[\deg 1]{f} & X \end{pmatrix} \right\} \Big/ \text{cobordism in domain and range.}$$

$L_n(K)$ depends only on π_1. $= L_n(\mathbb{Z}\pi)$.

Could take K any cpx with $\pi_1 K = \pi$!

$L_n(\mathbb{Z}\pi)$ has algebraic description in terms of quadratic forms, and automorphisms of forms.

Surgery exact sequence if $\eta(X) \neq \phi$.

$$\cdots \to L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(X^n) \to \eta(X) \to L_n(\mathbb{Z}\pi).$$

$$\begin{tikzcd} \nu_M \arrow[r, "b"] \arrow[d] & \xi \arrow[d] \\ \varphi : S^p \arrow[r, hook] & M^n \arrow[r, "f"', "\underset{\text{hof.}}{\simeq}"] & X \end{tikzcd}$$

$f \circ \varphi \simeq *$.

$S^p \times D^{n-p}$

$(M^n \xrightarrow{f} X)$ homotopy eq iff
1) $f_\#$ iso on π_1
2) $\tilde{f}_* : H_i(M, \Lambda) \xrightarrow{\simeq} H_i(X; \Lambda)$. $i \leq [\frac{n}{2}]$

$$\begin{tikzcd} 0 & K^{n-i}(f) \arrow[l] \arrow[d, "\simeq"', "\cap[M]"] & H^{n-i}(M;\Lambda) \arrow[l] \arrow[d, "\simeq"', "\cap[M]"] & H^{n-i}(X;\Lambda) \arrow[l, "f^*"'] \arrow[d, "\cap[X]"] & 0 \arrow[l] \\ 0 \arrow[r] & K_i(f) \arrow[r] & H_i(M;\Lambda) \arrow[r, "f_*"] & H_i(X;\Lambda) \arrow[r] & 0 \end{tikzcd}$$

For $p < [\frac{n}{2}]$ Attaching S^p killed S^{n-p-1} introduced.

So left with (1 or 2) middle dimension(s).

Kervaire-Milnor
$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \equiv 0 \\ 0 & 1 \\ \mathbb{Z}/2 & 2 \\ 0 & 3 \end{cases}$$

Swan cpx – $X(\pi)$.
finite Poincaré duality cpx

$\eta(X) \neq \phi$.

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] & G/O \arrow[r] & BO \arrow[r] & BG \arrow[r] & BG/O \\ & & & & X(\pi) \arrow[u, "\nu_X"] \arrow[ul, dashed] \arrow[ur, "\overline{\nu}_X"'] & \end{tikzcd}$$

$[X(\pi), BG/O] \rightarrowtail \bigoplus [X(\pi_p), BG/O]$.

by transfer.

Thomas-Wall I
Madsen-Thomas-Wall II, III.

p^2 + $2p$ conditions $\Rightarrow \pi$ acts freely on some S^{n-1} ($n = d(\pi)$ or $n = 2d(\pi)$)

– organize normal invariant $[X, G/O]$

– Dress induction

$$L_n(\mathbb{Z}\pi) \rightarrowtail \bigoplus_{\pi' \leq \pi} L_n(\mathbb{Z}\pi')$$

2-hyperelementary,

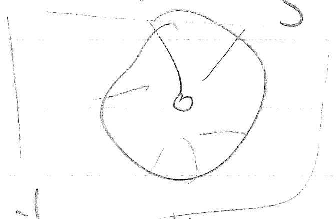
$$\eta(\xi) \xrightarrow{p\text{-elementary}} 0. \qquad \text{so } \eta(\xi) = 0$$

$\underline{p\text{-hyperelementary}}$: $1 \to C \to \pi' \to P \to 1$

C prime to p. P p-gp.

$\mathbb{R}^{8l+4}$. ⑦

Example S^{8l+3}



semifree action.

$\pi = Q(8p,q)$ period 4
dim 8 free rep.

$1 \to \mathbb{Z}/pq \to Q(8p,q) \to Q_8 \to 1$,
odd $p \neq q$.

$$Q_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \hookrightarrow \mathrm{aut}(\mathbb{Z}/pq).$$

$\zeta_r = e^{\frac{2\pi i}{r}}$

$$A = \mathbb{Z}[\eta_p, \eta_q]$$

$$F = \mathbb{Q}[\eta_p, \eta_q].$$

Thm π acts semifreely on $(\mathbb{R}^{8l+4}, 0)$ $l \geq 1$.

$$\Longleftrightarrow V(p,q) = (\eta_q - 2, \eta_p - 2) \in (A/p)^\times \times (A/q)^\times / \text{Squares}$$

lies in $\mathrm{Im}(\Phi_A|\ker \varphi_A)$

$$\Phi_A : F^{(2)} \longrightarrow (A/pq)^\times / \text{square}$$

$$V(p,q)$$

$$\varphi_A : F^{(2)} \longrightarrow (A/4)^\times / \text{squares}.$$

Ex: $pq < 2000$ (S. Bentzen.
$\exists!$, 42 pairs (p,q)
$(3,13)$, $(3,109)$
free actions on S^{8l+3}.
$(3,313)$, $(3,433)$, $(3,601)$, $(17,103)$.

$M \simeq \mathbb{C}P^3$

$\pi_2 M^6 = \mathbb{Z}$

$w_2 M = 0.$

$w(\mathbb{C}P^3) = (1+\alpha)^4 \quad |\alpha| = 2.$

$\exists\ \varphi : S^2 \times D^4 \longrightarrow M^6$ rep generator of $\pi_2 M^6$

Surgeries. $M' = (M \setminus \varphi(S^2 \times D^4)) \cup (D^3 \times S^3) = S^6$

embedded $S^3 \hookrightarrow S^6$.

$\text{knotted}(S^3 \subset S^6) = \mathbb{Z}$ Haefliger.

$\hat{M} = S^7 \circlearrowleft S^1 \longrightarrow M^6 \simeq \mathbb{C}P^\infty \to K(\mathbb{Z}, 2) = \mathbb{C}P^\infty.$

Gives inf many actions of S^1 on S^7. !