# Surgery theory & group actions I — Ian Hambleton, Aug 2011.

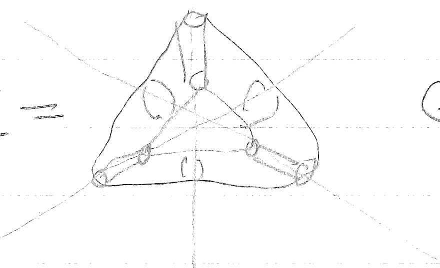
Beginning of geometric topology $\leftrightarrow$ covering spaces.

$$\mathrm{Aut}\begin{pmatrix}\tilde{X}\\ \downarrow\\ X\end{pmatrix} \longleftrightarrow \text{Subgroups of } \pi_1(X, x_0).$$

discrete group.

We saw instance this morning. $\begin{matrix}EG\\ \downarrow\\ BG\end{matrix}$

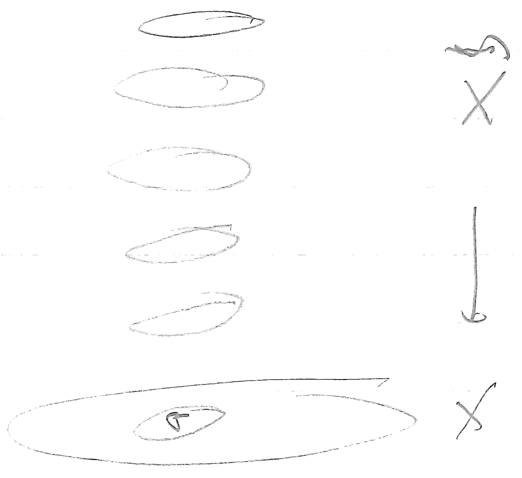
torus.

E.g. $\Sigma =$  $G = \Sigma_3$.

Quotient surface. $\Sigma/G$.

For infinite groups have to talk about properly discontinuous free actions. i.e.

$$\#\{\gamma \in G \mid \gamma C \cap C \neq \phi\} < \infty \quad \forall \; C \subseteq X \text{ compact.}$$



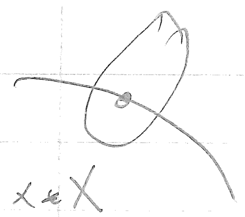
$\tilde{X}$ locally like $G \times$ open

" " $G \times D^n$

$X \leftarrow$ top $n$-manifold.

$\sigma = D^n$.

Def: A $G$-CW-cpx $X$ ($G$ cpt Lie, discrete) is a CW-cpx locally made up of cells $G/H \times D^n$, $H \leq G$ closed.

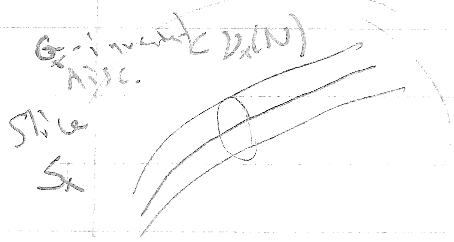


$x \in X$

$G_x = H$

$G/H \cong G/G_x \cong G \cdot x$ (orbit) through $x \in X$.

$M^n$ smooth $G$-manifold  $G$ cpt Lie gp.




$G x$ submanifolds.

tubular neighborhood $N(G \cdot x)$ has a slice structure.

$T_x M$ admits a linear $G_x$-action $\rho_x : G_x \to GL(T_x M)$ $= \mathbb{R}^n$ isotropy representation

$T_x M = T_x N \oplus \nu_x(N)$

Riemann $G$-invariant metric.

Thm (slice thm).

$$N(G_x) \cong \begin{array}{c} (S_x \times_H G) \\ \downarrow \\ G/H. \end{array}$$

Slice thm $\Rightarrow$ decomposing any compact $G$-manifold $(M, G)$ into principal bundles
$H \to G \to G/H$ closed.

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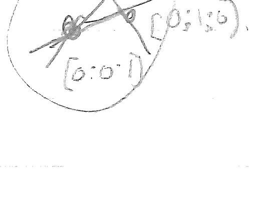
Example: $\mathbb{C}P^2$  $PGL_3(\mathbb{C}) = \{A \in GL_3(\mathbb{C})\} / \{\lambda I \mid \lambda \neq 0\}$.

interesting finite subgps.

$$t \longmapsto \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \qquad t^n = 1.$$




$\mathbb{Z}/n \longrightarrow PGL_3(\mathbb{C})$.  $t = e^{2\pi i/n}$

$a, b, c \in \mathbb{Z}$.

$$S^5 = S(\mathbb{C}^3) \circlearrowleft S^1$$

$$z \longmapsto \begin{pmatrix} z^a & 0 & 0 \\ 0 & z^b & 0 \\ 0 & 0 & z^c \end{pmatrix} \subset U(3)$$

$z \in S^1$. $\qquad a, b, c \in \mathbb{Z}$.

Montgomery – Yang problem (unsolved).

A pseudofree $S^1$ action on $S^5$ has $\leq 3$ exceptional orbits.

pseudofree $:\Leftrightarrow$ every orbit is a circle.
i.e. $G_x$ finite

exceptional orbits $:\Leftrightarrow$ $G_x \neq 1$.

Note $\#\{\text{exceptional orbits}\} < \infty$.
since isolated.

Note: The number 3 comes from linear model.
when $a, b, c$ are pairwise coprime.

$X = S^5/S^1$ = 4 dim orbifold $\mathbb{Q}$-homology $\mathbb{C}P^2$.

Surgery theory $\leftrightarrow$ classification of manifolds
(smooth, PL, TOP).

$\pi_1(M, x_0)$

Examples:

$G/H$, algebraic varieties.
(nonsingular).
→ hypersurfaces.
– complete intersections

– principal bundles.

– fibre bundles.

Group actions on manifolds

Goal understand symmetries of manifolds (with natural models)

- Symmetries $\overset{?}{\Rightarrow}$ restrictions on topology, .. curvature.

Question: Which compact Lie gps with $\dim G > 0$
can act freely on some $S^n$?

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**Problem:** Which f.t. gps can act on some $S^{n-1}$ by homeomorphisms?

~~Suppose X ft. CW-cpx $\pi_1(X,x_0) \cong \pi$, $\tilde{X} \simeq S^{n-1}$~~

a) by diffeomorphisms

b) $\exists$ ft. CW-cpx s.t. $\pi_1 X = \pi$ $\tilde{X} \simeq S^{n-1}$

c) $\exists$ X finite dim CW-cpx s.t.
$\pi_1 X = \pi$ $\tilde{X} \simeq S^{n-1}$

Suppose X ft. CW-cpx $\pi_1 X = \pi$ $\tilde{X} \simeq S^{n-1}$

We get exact seq

$$0 \to C_{n-1}(\tilde{X}) \to C_{n-1}(\tilde{X}) \to \qquad \to C_0(\tilde{X}) \to \mathbb{Z} \to 0$$

$C_i = C_i(\tilde{X})$ f.g. free $\mathbb{Z}\pi$-module.

Splicing $\rightsquigarrow$ $P_* \to \mathbb{Z} \to 0$ $\Rightarrow$ $H^*(\pi;\mathbb{Z})$ periodic
period $d \mid n$.

$$H^i(\pi;\mathbb{Z}) \cong H^{i+n}(\pi;\mathbb{Z}).$$

$g \in \mathrm{Ext}^n_{\mathbb{Z}\pi}(M,N)$ <
homological alg defn.
mult. ext defn.
$\mathrm{Hom}(\Omega^n M, N)/P\mathrm{Hom}(\Omega^n M, N)$.

$g$: $0 \to N \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$.
exact, $P_i$ projectives / $\mathbb{Z}\pi$.

$\Lambda = \mathbb{Z}\pi$. $\quad g \in \operatorname{Ext}^n_\Lambda(L, M)$

$\cup g : \operatorname{Ext}^i_\Lambda(M, N) \to \operatorname{Ext}^{n+i}_\Lambda(L, N)$

<u>Lemma 1:</u> If $g$ rep by proj $Q_*$

$\Rightarrow$ $\underbrace{\cup_g \text{ iso } i>0 \text{ epi } i=0.}_{(*)\ \forall N.}$

<u>Thm</u> (Roggenkamp?, C.T.C. Wall).

If $g \in \operatorname{Ext}^n(L, M)$ has the property $(*)$ $\forall N$.
then $g$ is represented by a projective resolution.

<u>Thm</u> (Swan, 1960). If $\pi$ is a finite gp with periodic cohomology, then $\exists$ periodic proj resolution. $\boxed{n\text{ even}}$

$0 \to \mathbb{Z} \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0.$ (period $n$).

$P_i$ finitely generated proj mod / $\mathbb{Z}\pi$.

$$\Theta(g) = \sum_{i=0}^{n-1} (-1)^i [P_i] \in \tilde{K}_0(\mathbb{Z}\pi) = \left\{ \begin{array}{l} \text{grothendieck gp of} \\ \text{finitely gen proj } \mathbb{Z}\pi\text{-mod} \end{array} \right\}.$$

<u>Thm</u> (Swan) $\tilde{K}_0(\mathbb{Z}\pi)$ finite ab gp. / free $= 0$.

If you splice $g_1$ and $g_2$

$$\Theta(g_1 \# g_2) = \Theta(g_1) + (-1)^n \Theta(g_2).$$

In particular $\Theta(g^r) = r\,\Theta(g)$ $\quad$ ($n$ even)

So can choose

$r = \exp(\tilde{K}_0)$ get $\Theta(g^r) = 0$.

~~Thm~~ (Milnor) This can be geometrically realized!

We hence get: (in fact f.t. dominated)

**Thm** (Swan) For any $\pi$ fi. grp w/ periodic coham period $n$ $\exists\, X(\pi)$ finitely ~~dominated~~ dimensional CW-cpx with

$$\pi_1(X(\pi)) \cong \pi \qquad \tilde{X}(\pi) \simeq S^{n-1}.$$

By finite joins, can get a finite $X(\pi)$ with $\tilde{X}(\pi) \simeq S^{rn-1}$.

Another answer of hom alg: $\pi$ periodic $\Leftrightarrow$ Every subgp of order $p^2$ is cyclic.

$$\text{So} \quad \pi_p = \begin{cases} \text{cyclic} \\ \text{gen quaternionic } (p=2). \end{cases}$$

Big thm:

**Thm** (Madsen, Thomas, Wall) $\pi$ acts finitely top on some $S^{n-1}$ or $S^{2n-1}$ $\Leftrightarrow$ $p^2$-conditions, $2p$-conditions (Milnor '57)

$n = \text{period}(\pi)$.

Example: $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$. acts freely, not orthogonally.