

$X \rightarrow G$ -h. equivalence
 $\rightarrow hG$ -hom. eq.

Homotopical group actions III

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$$X \rightsquigarrow X^P \quad X^{hP}$$

From Alejandro's lecture

$H_{h\mathbb{Z}/p}^*(X, X^{\mathbb{Z}/p})$ is finite since $\mathbb{Z}/p$ acts freely on $X - X^{\mathbb{Z}/p}$

If X -space $\rightsquigarrow H_*(X; \mathbb{F}_p)$

Is this a ^{complete} model for X ? NO in general, but YES if X is p -complete, $\tilde{H}^*(X, \mathbb{F}_p) \simeq \tilde{H}^*(S^\infty, \mathbb{F}_p)$

(Completion: Bousfield-Kan completion
 X -complete & $H_*(X, \mathbb{F}_p) = 0 \Rightarrow X$ simply connected)

Def: $\mathbb{F}_p$ -localization: $X \mapsto X_p^\wedge$ with the property that $X \rightarrow X_p^\wedge$ is isom. in $\mathbb{F}_p$ -homology & it is terminal w.r.t. this property.

$$\begin{array}{ccc} X & \longrightarrow & X_p^\wedge \\ & \searrow \scriptstyle \text{iso on } \mathbb{F}_p & \nearrow \\ & \mathcal{O} & \end{array}$$

$X = \mathbb{F}_p$ -local: $\forall f: Y \rightarrow Z$ $H_*(-, \mathbb{F}_p)$ -isom. then the induced map

$[Z, X] \xrightarrow{\simeq} [Y, X]$ is a bijection.

$$\begin{array}{ccc} X & \longrightarrow & X_p^\wedge \\ \downarrow & \mathcal{O} & \downarrow \\ Y & \xrightarrow{\quad} & Z \\ \uparrow & \mathcal{O} & \uparrow \\ 1 & & 1 \\ \text{local} & & \end{array}$$

Def: X is p -good if $X \xrightarrow{\mathbb{F}_p} \mathbb{F}_p X$ is $\mathbb{F}_p$ -isom. $\mathbb{B}\mathbb{K}$ -space

p -good $\Rightarrow X \rightarrow \mathbb{F}_p X$ is $\mathbb{F}_p$ -localization

Def: p -complete means $X \rightarrow \mathbb{F}_p X$ is a homotopy equivalence

• $\mathbb{F}_p$ -complete $\Leftrightarrow \mathbb{F}_p$ -local + $\mathbb{F}_p$ -good

• $H_*(Y; \mathbb{F}_p)$ finite type $\Rightarrow$ Sullivan $\mathbb{F}_p$ -completion $\Leftrightarrow$ BK $\mathbb{F}_p$ -completion

augmented $\mathbb{F}_p$ -field every chain complex splits

$$X \rightsquigarrow \tilde{C}_*(X; \mathbb{F}_p) \xleftarrow{\text{trivial}} H_*() \quad \left. \begin{array}{l} \text{trivial} \\ G\text{-action} \end{array} \right\}$$

$$\times: G/p \hookrightarrow \tilde{C}_*(X^p, \mathbb{F}_p)$$

$$O_p(G)^{op} \hookrightarrow \text{Ch}_*(\mathbb{F}_p\text{-modules})$$

chain complexes of $\mathbb{F}_p$ -modules

$$\times \in \text{Ch}_*(\mathbb{F}_p O_p(G)^{op})\text{-modules} \quad \leftarrow \text{not here}$$

Thm (G-Smith) Compact model thm:

•) If G is a finite group, X = finite G -CW-complex

$\times: G/p \rightarrow \tilde{C}_*(X^p, \mathbb{F}_p)$ is a quasi isom. to a finite

chain complex of projective $\mathbb{F}_p O_p(G)^{op}$ -modules (minimal model which is unique up to isomorphism)

•) If G = finite group, X = G -space s.t. $H^*(X, \mathbb{F}_p)$ = finite $\Rightarrow$

$\times: G/p \rightarrow \tilde{C}_*(X^p, \mathbb{F}_p)$ is a quasi isom. to a finite chain complex of proj. $\mathbb{F}_p O_p(G)^{op}$ -modules.

Ex: $X = S^0$ $\tilde{C}_*(S^0, \mathbb{F}_p) = C_*(pt, \mathbb{F}_p)$

trivial action $G/p \hookrightarrow \mathbb{F}_p$ (in degree 0)

$\mathbb{F}_p$ has a finite projective resolution in $\mathbb{F}_p O_p(G)^{op}$ -modules

$$\lim_{\leftarrow}^{*} \mathbb{F}_p O_p(G)^{op} F = \bigoplus_{\mathbb{F}_p O_p(G)^{op}}^* (\mathbb{F}_p, F)$$

$$H_0^*(|Sp(G)|; F)$$

higher limits for any F vanish in high degrees.

(Jackowski - McClure - Oliver '88)

Furthermore = Fact abouts this resolution translate into facts about higher limits.

$(C_*(EO_p(G); \mathbb{F}_p) = \text{canonical projective resolution, Infinite!!})$
but equivalent to a finite model (JMO)

Projectives:

$$P_1 \quad K \rightrightarrows K[G/C_2]$$

$$P_2 \quad 0 \rightarrow K[G/C_3]$$

$$P_3 \quad 0 \rightarrow S^t$$

"projective"

$$(\mathbb{F}_2 \Sigma_3 = M_2(\mathbb{F}_2) \oplus P_{\mathbb{F}_2})$$

"st"

$$0 \rightarrow st \rightarrow st \oplus st$$

$$K \rightrightarrows K[G/C_2] \rightarrow \text{projective resolution of the functor}$$

$$K \rightarrow K$$

$$0 \rightarrow P_2$$

$$\downarrow$$

$$\vdots$$

! minimal model for the sphere

$$\downarrow$$

$$P_3 + P_2$$

$$\downarrow$$

$$P_1$$

Ex. session

G = finite
graph
 p = prime

$$ch_*(O_p(G)^{op} \rightarrow \mathbb{F}_p\text{-module})$$

X = finite G -CW-complex

$$\mathcal{O}_H \rightarrow \tilde{C}_*(X^H; \mathbb{F}_p)$$

the $\mathbb{X} : O_p(G)^{op} \rightarrow \tilde{C}_*(X^H; \mathbb{F}_p) \in (ch_*(O_p(G)^{op} \rightarrow \mathbb{F}_p\text{-mod}))$

$IP \in \dots$, $P = P_p$, $IP = \text{finite}$, $P_i = \text{projective}$

Why $\mathcal{O}$ modules are in this form?

$$\mathcal{O} = O_p(G)^{op} \rightarrow \mathbb{F}_p\text{-mod}$$

$$\mathbb{Z}[G/H] \text{ and } \mathbb{F}_p[G/H] \neq 0 \text{ and } \mathbb{F}_p[G/H] \neq 0 \text{ and } \mathbb{F}_p[G/H] \neq 0$$

$$\rightarrow \mathbb{F}_p[G/H] \text{ is a module over } \mathbb{F}_p[G/H] \text{ and } \mathbb{F}_p[G/H] \neq 0$$

Homological algebra in functor categories

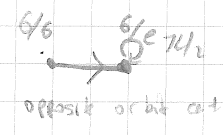
$\mathbb{F}_p \mathcal{O}_p(\mathcal{G}^{\text{op}})$ -modules

Simple objects $\leftrightarrow$ Functors which is non-zero on only one $\mathcal{G}/p$
 simple $\mathbb{F}_p N_{\mathcal{G}}(P)/p$ -module S
 $(\mathcal{G}/p, S)$ has no proper subfunctor

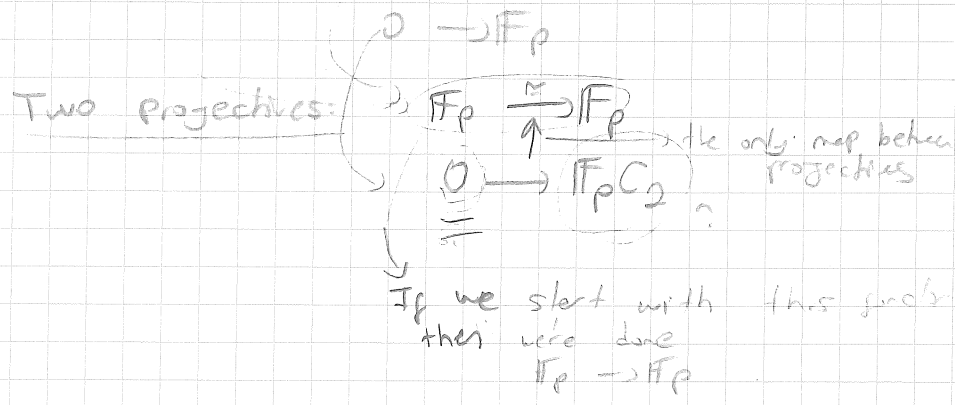
(indecomposable)
 $\rightarrow$ Projective objects: $S = \text{simple}$ if $\mathbb{F}_p N_{\mathcal{G}}(P)/p\text{-module} \leadsto P_S$ projective cover

$$P_{(\mathcal{G}/p, S)}(\mathcal{G}/0) = \mathbb{F}_p \text{Map}_{\mathbb{F}_p N_{\mathcal{G}}(P)/p}(\mathcal{G}/0, \mathcal{G}/p) \otimes P_S$$

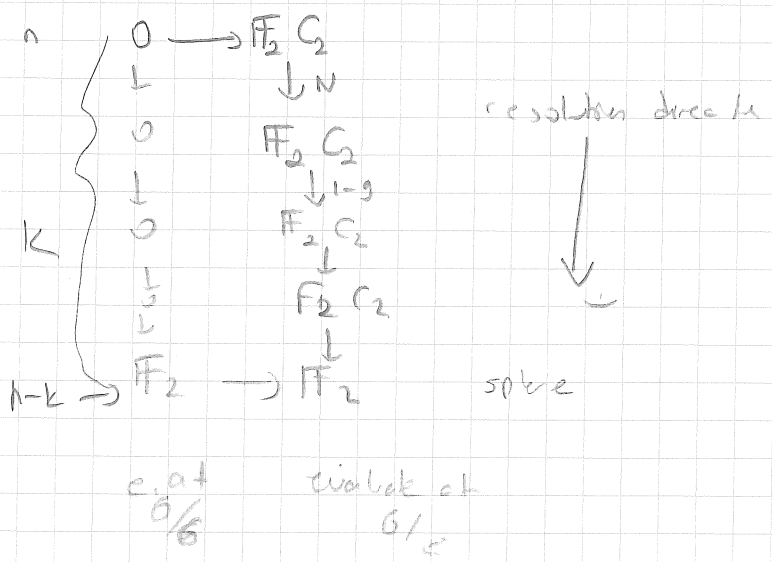
Ex 1 $\mathcal{G} = \mathbb{Z}/2$, $p=2$



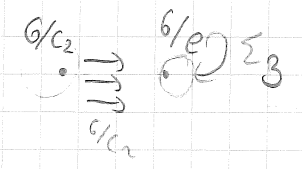
Two simples: $\mathbb{F}_p \rightarrow 0$



Smith theory:



Ex 2 Σ_3 , $p=2$ $\mathcal{O}_p(\mathcal{G})$



$L = \mathbb{F}_p$
 Simple functors: $k \rightarrow 0$
 $0 \rightarrow k$
 $0 \rightarrow St \rightarrow 1^k$

Simple $\mathbb{F}_2 \Sigma_3$ -modules: k

$$\Sigma_3 = GL_2(\mathbb{F}_2) \quad St = V = \{ \langle x_i \in \mathbb{F}_2^3 \mid \sum x_i = 0 \}$$