$X \to G$-h. equivalence
$\searrow hG$-hom. eq.

$X \rightsquigarrow X^P \quad X^{hP}$

Homotopical group actions III

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From Alejandro's lecture

$H^*_{h\mathbb{Z}/p}(X, X^{\mathbb{Z}/p})$ is finite since $\mathbb{Z}/p$ acts freely on $X - X^{\mathbb{Z}/p}$

If X = space $\rightsquigarrow H_*(X; \mathbb{F}_p)$

Is this a complete model for X? NO in general, but YES if X is p-complete. $\tilde{H}^*(X, \mathbb{F}_p) \simeq \tilde{H}^*(S^n, \mathbb{F}_p)$

(Completion: Bousf. - Kan completion
X = complete & $H_1(X, \mathbb{F}_p) = 0 \Rightarrow X$ = simply connected
$S^{n\wedge}_p \xrightarrow{\simeq} X$)

Def: $\mathbb{F}_p$-localization: $X \longmapsto X^\wedge_p$ with the property that $X \to X^\wedge_p$ is isom. in $\mathbb{F}_p$-homology & it is terminal wrt this property.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\text{iso on } \mathbb{F}_p"'] & X^\wedge_p \\ Y \arrow[ur, dashed] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

X = $\mathbb{F}_p$-local: $\forall f: Y \to Z$ $H_*(-, \mathbb{F}_p)$-ism. then the induced map
$[Z, X] \xrightarrow{\simeq} [Y, X]$ is a bijection.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^\wedge_p \arrow[dl, dashed, "\exists !"] \\ Y & \end{tikzcd}$$

local

Def: X = p-good if $X \to \mathbb{F}_{p\infty} X$ is $\mathbb{F}_p$-isom
$\underbrace{\mathbb{F}_{p\infty} X}_{\text{B-K-space}}$

- p-good $\Rightarrow X \to \mathbb{F}_{p\infty} X$ is $\mathbb{F}_p$-localization

Def: p-complete means $X \to \mathbb{F}_{p\infty} X$ is a homotopy equivalence.

- $\mathbb{F}_p$-complete $\Leftrightarrow$ $\mathbb{F}_p$-local + $\mathbb{F}_p$-good
- $H_*(X;\mathbb{F}_p)$ finite type $\Rightarrow$ Sullivan $\mathbb{F}_p$-completion $\Leftrightarrow$ BK $\mathbb{F}_p$-completion

augmented $\quad$ $\mathbb{F}_p$-field every chain complex splits

$X \rightsquigarrow \tilde{C}_*(X;\mathbb{F}_p) \leftrightarrow H_*(\ \ )$ $\quad \leftarrow$ trivial G-action

$\mathcal{X}$: $G/P \longmapsto \tilde{C}_*(X^P;\mathbb{F}_p)$

$\mathcal{O}_p(G)^{op} \longmapsto Ch_*(\mathbb{F}_p\text{-modules})$

chain complexes of $\mathbb{F}_p$-modules

$\mathcal{X} \in Ch_*(\mathbb{F}_p\mathcal{O}_p(G)^{op}\text{-modules})$ $\quad$ not here

Thm (G-Smith) Compact model thm:

- If G is a finite group, X = finite G-CW-complex

Tim's thm

$\mathcal{X}: G/P \longrightarrow \tilde{C}_*(X^P;\mathbb{F}_q)$ is a quasi isom. to a finite chain complex of projective $\mathbb{F}_p\mathcal{O}_p(G)^{op}$-modules (minimal models which is unique up to isomorphism)

- If G = finite group, X = G-space s.t. $H^*(X,\mathbb{F}_p)$ = finite $\Rightarrow$ $\mathcal{X}: G/P \longrightarrow \tilde{C}_*(X^{hP},\mathbb{F}_p)$ is a quasi isom. to a finite chain complex of proj. $\mathbb{F}_p\mathcal{O}_p(G)^{op}$-modules.

Ex: $X = S^0$ $\quad$ $= \tilde{C}_*(S^0;\mathbb{F}_p) = C_*(pt,\mathbb{F}_p)$

trivial action $\quad G/P \longmapsto \mathbb{F}_p$ (in degree 0)

$\mathbb{F}_p$ has a finite projective resolution in $\mathbb{F}_p\mathcal{O}_p(G)^{op}$-modules

$$\lim^*_{\mathcal{O}_p(G)^{op}} F = \mathrm{Ext}^*_{\mathbb{F}_p\mathcal{O}_p(G)^{op}}(\mathbb{F}_p, F)$$

$$\| \\ H^*_G(|S_p(G)|; F)$$

higher limits for any F vanish in high degrees. (Jackowski - McClure - Oliver 98)

Furthermore: Fact abouts this resolution translate into facts about higher limits.

($C_*(E\mathcal{O}_p(G);\mathbb{F}_p)$ = canonical projective resolution. Infinite!! but hequivalent to a finite model (JMO))

Projectives:

$P_1 \quad k \rightrightarrows k[G/C_2]$

$P_2 \quad 0 \longrightarrow k[G/C_3]$

$P_3 \quad 0 \longrightarrow St$
$\qquad\qquad\qquad\quad \shortparallel$
$\qquad\qquad\quad$ projective

$\mathbb{F}_2 \Sigma_3 = M_2(\mathbb{F}_2) \oplus P_{\mathbb{F}_2}$
$\qquad\qquad \shortparallel$
$\qquad\quad St \oplus St$

$$\begin{array}{ccc} 0 & \to & St \\ & & \downarrow \\ k & \rightrightarrows & k(G/C_2) \\ & & \downarrow \\ k & \longrightarrow & k \end{array}$$

$\to$ projective resolution of the functor

$$\begin{array}{c} 0 \longrightarrow P_2 \\ \downarrow \\ \vdots \\ \downarrow \\ P_3 + P_2 \\ \downarrow \\ P_1 \end{array}$$

! minimal model for the sphere

Ex. session:

G = finite group
p = prime

$ch_*(O_p(G)^{op} \longrightarrow \mathbb{F}_p\text{-module})$

X = f.t. G-CW-complex

$G/H \longrightarrow \tilde{C}_*(X^H; \mathbb{F}_p)$

thm: $\cancel{X} : O_p(G)^{op} \longrightarrow \tilde{C}_*(X^?, \mathbb{F}_p) \in Ch_*(O_p(G)^{op} \to \mathbb{F}_p\text{-mod})$

$\wr\wr$

$\mathbb{P} \in$, $\mathbb{P} = P_*$, $\mathbb{P}$ = finite P_i = projective $: O_p(G)^{op} \to \mathbb{F}_p$-module

Why simples are in this form?

$\forall \; F : O_p(G)^{op} \to \mathbb{F}_p\text{-mod}$

$\exists\, G/H$ s.t. $F(G/H) \neq 0$ and [illegible]

[illegible]

→ Homological algebra in functor categories
$\mathbb{F}_p \mathcal{O}_p(G)^{op}$-modules

Simple objects $\longleftrightarrow$ Functors which is non-zero on only one G/P
simple $\mathbb{F}_p N_G(P)/P$-module S
has no proper subfunctor

$(G/P\,,\,S)$

(indecomposable) Projective objects : S = simple if $\mathbb{F}_p N_G(P)/P$-module $\rightsquigarrow P_S$ projective cover

$$P_{(G/P,S)}(G/Q) = \mathbb{F}_p \,\mathrm{Map}_G(G/Q,\, G/P) \otimes_{\mathbb{F}_p N_G(P)/P} P_S$$

- ~~Ex~~ Ex ① $G = \mathbb{Z}/2$, $p=2$

$G/G \longrightarrow G/e \circlearrowleft \mathbb{Z}/2$

opposite orbit cat

Two simples : $\mathbb{F}_p \to 0$

$0 \to \mathbb{F}_p$

Two projectives: $\mathbb{F}_p \xrightarrow{\cong} \mathbb{F}_p$ ← the only map between projectives

$0 \longrightarrow \mathbb{F}_p C_2$?

If we start with this functor then we're done
$\mathbb{F}_p \to \mathbb{F}_p$

<u>Smith theory:</u>

$$\begin{array}{lccc}
n & 0 & \longrightarrow & \mathbb{F}_2 C_2 \\
 & \downarrow & & \downarrow N \\
 & 0 & & \mathbb{F}_2 C_2 \\
 & \downarrow & & \downarrow 1-g \\
k & 0 & & \mathbb{F}_2 C_2 \\
 & \downarrow & & \downarrow \\
 & \downarrow & & \mathbb{F}_2 C_2 \\
 & \downarrow & & \downarrow \\
n-k \to & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2
\end{array}$$

resolution direction ↓

sphere

e. at G/G evaluate at G/e

<u>Ex ②</u> . Σ_3 , $p=2$ $\mathcal{O}_p(G)$ $k = \mathbb{F}_p$

$G/C_2 \rightrightarrows G/e \circlearrowleft \Sigma_3$

G/C_2

<u>Simple functors:</u> $k \longrightarrow 0$

$0 \longrightarrow k$

$0 \longrightarrow St$ ~~~ ... ~~~

<u>Simple $\mathbb{F}_2\Sigma_3$-modules:</u> k

$\Sigma_3 = GL_2(\mathbb{F}_2)$ $St = V = \{ (x_i) \in \mathbb{F}_2^3 \mid \sum x_i = 0 \}$