Homotopical group actions II

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Smith theory:

Thm (P.A. Smith): $P = \mathbb{Z}/p$, X = finite P-CW-complex, 1) $H_*(X, \mathbb{F}_p) \simeq \mathbb{F}_p$
$\Rightarrow H_*(X^P, \mathbb{F}_p) \simeq \mathbb{F}_p$

2) $H_*(X) \simeq H_*(S^n) \Rightarrow H_*(X^P) \simeq H_*(S^k)$, $k \leq n$ ← with $\mathbb{F}_p$ coeff

(Just like the linear case!) (linear representations)

General question: Try to determine X^P or X^{hP} from X
(on mod p homology on spaces "at a prime p")

(as a functor of X as a P-space up to hP-equivalence as functor $H_*(X)$)

X = finite

1) says if X is $\mathbb{F}_p$-acyclic $\Rightarrow X^P$ $\mathbb{F}_p$-acyclic

$$X \xrightarrow[\mathbb{F}_p - hP\text{-equivalence}]{} *$$

true for elementary abelian groups

Thm (Localization thm, Borel, Quillen, W.Y. Hsiang): $V = \mathbb{Z}/p$, X = finite V-CW-complex

S = multiplicative set generated by the non-trivial elements in $H^1(\mathbb{Z}/2; \mathbb{F}_2)$
$p=2$

$$H^*_{hV}(X) := H^*(X_{hV}) \longrightarrow H^*_{hV}(X^V) = H^*(V) \otimes H^*(X^V)$$ modules over $H^*(V)$

Borel equiv. coh

how far this is being isomorphism

$$S^{-1} H^*_{hV}(X) \xrightarrow{\simeq} S^{-1} H^*_{hV}(X^V)$$

p = odd
$H^*(\mathbb{Z}/p, \mathbb{F}_p) = \mathbb{F}_p[u] \otimes \Lambda(v)$

$S = (u^i)$

(Like what Alejandro proved this morning)

Proof: 1) Use $H^*_{hV}(X, X^V)$ is finite

2) Both RHS & LHS equiv. cohomology theories
Check for $(G/H \times D^n, G/H \times S^{n-1})$

trivial cells (D^n, S^{n-1}) ✓

V-cells $(V \times D^n, V \times S^{n-1})$ ✓ (both LHS & RHS = 0)

Theorem (Dwyer–Wilkerson 88): X = finite V-CW-complex, V = elementary abelian group.

$$H^*_{hV}(X^V) \xrightarrow{\simeq} U_n(S^{-1}H^*_{hV}(X^V)) \xleftarrow{\simeq} U_n(S^{-1}H^*_{hV}(X))$$

$H^*_{hV}(X^V) = H^*(V) \otimes H^*(X^V)$

U_n ← take largest unstable module ?

$$U_n(S^{-1}H^*(V) \otimes H^*(X^V))$$

In particular, $H^*(X^V) = \mathbb{F}_p \otimes_{H^*(V)} U_n(S^{-1}H^*_{hV}(X))$

<u>Definition</u>: $U_n(M) = \{x \in M \mid Sq^I x = 0 \text{ if } \mathrm{excess}(I) > |x|\}$

$$Sq^I = Sq^{i_1} \cdots Sq^{i_k}$$

$$\mathrm{excess}(I) = \sum_\rho (i_\rho - 2i_{\rho+1})$$

<u>Proof</u>: We have to prove if M is unstable module $R = H^*(V)$, $S = H^1(V) \setminus \{0\}$

$R \otimes M \to M$ R-module over the Steenrod algebra

$$M \xrightarrow{\simeq} U_n(S^{-1}R \otimes M)$$

"things with denominators will never be unstable"

$$P_\xi\left(\frac{x}{s}\right) = P_\xi(x)\, P_\xi(s)^{-1}$$

$$P_\xi(s)^{-1} = (s + Sq^1 s\, \xi + Sq^2 s\, \xi^2 + \cdots)^{-1}$$

<u>Thm (Lannes, Dwyer, Wilkerson)</u>: $H^*(X)$ finite   (X not necessarily finite)

Then $H^*(X^{\wedge hV}_p) = \mathbb{F}_p \otimes_{H^*(V)} U_n(S^{-1}H^*_{hV}(X))$.

+ small technical assumption (e.g. X^V simply connected)

$$H^*(X^{hV}) = H^*(X^V)$$

⇒ Sullivan conjecture

<u>Thm</u>: $X^{V\wedge}_p \xrightarrow{\simeq} X^{\wedge hV}_p$ when X = finite V-CW-complex.

(Miller, Lannes, Carlsson) (Dwyer–Miller–Neisendorfer)

$\rightarrow$ P = finite p-group X = finite P-CW-complex

$$X_p^{P\wedge} \xrightarrow{\simeq} X_p^{\wedge\, hP}$$

Lannes: $\overline{X}$ = arbitrary V-CW-complex (+ small technical assump.)

$$\underset{H^*(V)}{H^*(X^{hV}) = \mathbb{F}_p \otimes} \operatorname{Fix}(H^*_{hV}(X))$$

- V acts trivially, $X^{hV} = \operatorname{map}(BV, X)$

$$\Rightarrow H^*(\operatorname{map}(BV, X)) = \ldots (H^*(X))$$

computable as a functor of $H^*(X)$

In particular, X = finite $\operatorname{map}(BV, X) \xrightarrow{\simeq} X$

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$G \rightsquigarrow BG$ $\quad H^*(G, \mathbb{F}_p)$

$EG \times_G X$ $\quad$ P. Smith theory
localization thm.

What can this approach say about free actions?

<u>Case:</u> $X = S^n$ with a free G-action

p^2-condition: • If G acts freely on X, then every abelian subgroup is cyclic in G.
$\Updownarrow$
• $\mathbb{Z}/p \times \mathbb{Z}/p$ does not occur as a subgroup of G

$\boxed{\text{homological}}$

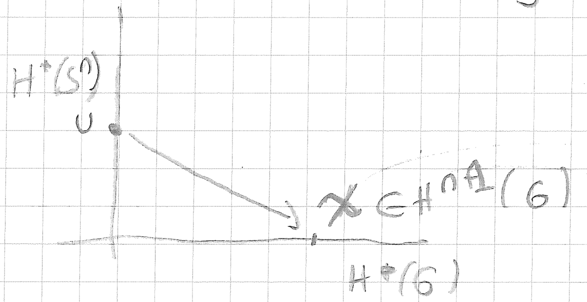
$2p$-condition (Milnor): If G acts freely on X then every involution in G is central

$\boxed{\text{geometric condition}}$

$2p + p^2 = MTW$

• G acts freely on <u>some</u> such $X \Leftrightarrow G$ satisfies p^2 & $2p$ for all p

$\to X = S^n$ G acts freely $EG \times_G X \simeq X/G$ preserving orientation



Spectral sequence generates to the so-called Gysin sequence

Euler class of the action

$$\cdots \to H^{i+1}(X/G) \to \boxed{H^i(G) \xrightarrow{\cup x} H^{i+n+1}(G)} \to H^{i+n+1}(X/G) \to \cdots$$

Note that $\cup x$ is going to induce an isomorphism $\Rightarrow H^*(G)$ - periodic of period $n+1$. (x is a periodicity generator)

Note this holds for all subgroups of G so they all have periodic cohomology.

$H^*(\mathbb{Z}/p \times \mathbb{Z}/p)$ is not periodic $\Rightarrow \mathbb{Z}/p \times \mathbb{Z}/p \not\subset G$.

Q_8: $x \in H^4(Q_8, \mathbb{Z}) \cong \mathbb{Z}/8$

$$H^i(Q_8, \mathbb{Z}) \cong \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i \equiv 2 \mod 4 \\ \mathbb{Z}/8 & i \equiv 0 \mod 4 \end{cases}$$