$\pi = \mathbb{Z}/7 \rtimes \mathbb{Z}/3 \to$ period 6

act freely on a sphere not $\underbrace{\text{freely and orthogonally}}_{\text{no conditions}}$ (not on a sphere of a representation?)

= Homotopical group actions I =

16.08.2011
J. Grodal

<u>Basic setup</u>: G = finite group

objects: X admit the structure of a G-CW-complex

cells: $G/H \times D^n$

morphisms: G-maps $f: X \to Y$ $\quad f(gx) = g f(x)$

G-homotopic if $\exists$ $F: X \times I \to Y$ s.t. $f = F(-,0)$ & $f' = F(-,1)$

G-equivariant (with trivial action on I)

<u>Def</u>: • A G-map $f: X \to Y$ is called G-equivalence if $\exists^{G\text{-map}}$ $f': Y \to X$ s.t. $f \circ f'$ and $f' \circ f$ are G-homotopic to id.

<u>Note</u>: G-cellular approximation holds

• A G-map $f: X \to Y$ is called an hG-equivalence (or "pseudo" weak G-equivalence) if $f: X \to Y$ is a non-equiv equivalence (i.e. $\exists$ (non-G)-map $f': Y \to X$ s.t. $f': Y \to X$ s.t. $f' \circ f$, $f' \circ f$ are (non-equiv) homotopic to id.

<u>Thm</u>: (Equiv. Whitehead thm) A G-map $f: X \to Y$ is a G-homotopy equivalence $\iff \forall H \leq G$ $f^H: X^H \to Y^H$ is a non-equivariant equivalence

$\iff \pi_*(X^H) \xrightarrow{\cong} \pi_*(Y^H) \quad \forall H \leq G.$

<u>Proof</u>: "$\Rightarrow$" follows from def.

"$\Leftarrow$" induction on equivariant cells. $X^G \rightleftarrows Y^G$

<u>Notes</u>: • Warning! Fixed points are not invariant under hG-equivalence

$EG \times X \xrightarrow[\text{proj.}]{} X$ hG-equivalence but $(EG \times X)^H = \emptyset$ $\forall 1 \lneq H \leq G$.

• View $\{G\text{-spaces}\} / hG\text{-equivalence} \longleftrightarrow \text{Functors}(\mathcal{G}, \text{Spaces}) / \text{objectwise h. equivalence}$

Def: The **Orbit category** $\mathcal{O}(G)$ is a category objects G/H $H \le G$

morphisms G-maps

$$\mathrm{Mor}_{\mathcal{O}(G)}(G/H, G/K) = \{g \in G \mid H^g \le K\}/K = (G/K)^H$$

p-orbit category = full subcategory obj G/P, $P = p$-subgroup

Basic observation: $\Phi: \{G\text{-spaces}\} \longrightarrow \mathrm{Fun}(\mathcal{O}(G)^{op}, \text{Spaces})$

$$X \longrightarrow (G/H \longrightarrow X^H$$

$$f: G/H \to G/K$$
$$H \to gK$$

Functor $G/H \to \mathrm{Hom}_G(G/H, X)$

Thm (Elmendorf): The functor Φ gives a 1-1 correspondence between

$$\{G\text{-spaces}\}/G\text{-eq} \longleftrightarrow \mathrm{Fun}(\mathcal{O}(G)^{op}, \text{Spaces}\}/\text{pointwise equivalence}$$

In fact: comes from Quillen eq of model categories

Ex $G = \mathbb{Z}/p$ $\quad \mathcal{O}(G)^{op}$ $\quad$ $G/G \longrightarrow G/e \circlearrowleft G$ (opposite category) with trivial G-action on it!

$\mathbb{Z}/p$-homotopy types $\quad$ equiv map $Y_0 \xrightarrow{f} Y_1^{\mathbb{Z}/p}$

(non equiv. map $Y_0 \xrightarrow{f} Y_1^{\mathbb{Z}/p}$)

Ex: Note: Any homotopy can occur as Y_0 given Y_1 (if $Y_1^{\mathbb{Z}/p} \neq \emptyset$)

(No "Smith theory")

$\mathbb{Z}/p$ - homotopy equiv. in some sense reduced to $h\mathbb{Z}/p$ - equiv.

Proof of Elemendorf's thm: Baby case $\mathbb{Z}/p$

X = homotopy pushout
$$\left(\begin{array}{ccc} EG \times Y_0 & \longrightarrow & Y_0 \\ \downarrow & & \\ EG \times Y_1 & & \end{array}\right)$$

Non-equivariantly $X \underset{hG}{\simeq} Y_1$ $\quad (EG \simeq *)$

$$X^{\mathbb{Z}/p} \simeq Y_0$$

$$\Phi(X) \longrightarrow (Y_0 \to Y_1^{\mathbb{Z}/p})$$

<u>Proof of Elmendorf's thm:</u>
X = double mapping cylinder
$$\begin{pmatrix} EG \times Y_0 \to Y_0 \\ \downarrow \\ EG \times Y_1 \end{pmatrix}$$

Want to make space from fixed points

$C(F) := \operatorname{hocolim}_{E\mathcal{O}(G)} F$ $\qquad C: \mathrm{Fun}(\mathcal{O}(G)^{op}, \{\text{spaces}\}) / \text{pointwise equiv.} \longrightarrow \{G\text{-spaces}\} / G\text{-equiv.}$

$E\mathcal{O}(G) = G/e \downarrow \mathcal{O}(G)$ $\left[\begin{array}{l} \text{objects } G/e \to G/H \\ \text{morphisms } \begin{array}{ccc} G/e & & \\ \downarrow & \searrow & \\ G/H & \to & G/H' \end{array} \end{array}\right]$

$$E\mathcal{O}(G)^{op} \longrightarrow \mathcal{O}(G)^{op} \xrightarrow{F} \text{Spaces}$$

<u>Claim:</u> $\Phi C \to id$ is an equivalence

$$X \longmapsto C(\Phi(X))$$
$$\downarrow$$
$$X$$

$C(F) := \operatorname{hocolim}_{E\mathcal{O}(G)} F = B(E, \mathcal{O}(G)^{op}, F)$ $\qquad \begin{pmatrix} E : \mathcal{O}(G) \to G\text{-spaces} \\ G/H \mapsto G/H \end{pmatrix}$

simplicial space
n-simplices

$$\left(G/e \to G/H_0 \to G/H_1 \to \cdots \to G/H_n, \; x \in F(G/H_n) \right)$$

$$\Phi C(F)(G/H) \longrightarrow F(G/H)$$

$$\left(\operatorname{hocolim}_{E\mathcal{O}(G)} F\right)^H = \operatorname{hocolim}_{(E\mathcal{O}(G)^H)^{op}} F \simeq \operatorname{hocolim}_{(S(G)_{\geq H})^{op}} F \xrightarrow{\simeq} F(G/H)$$

posets of subgroups in G containing H

So G-homotopy type $\rightsquigarrow$ hN_H/H-homotopy types $\forall H \leq G$

+ compatibility.

What about hG-equivalence?

<u>Invariants of hG-equivalence</u>: • <u>Borel construction</u>: $(X \times EG)/G = X_{hG}$

(homotopy orbit space)

$$\begin{tikzcd} G \arrow[r, hook] & EG \arrow[d] \\ & BG \end{tikzcd}$$

• homotopy fixed points:

$$X^{hG} = \mathrm{map}_G(EG, X) = \mathrm{map}(EG, X)^G$$

Note $EG \to *$ gives a map $X^G \to X^{hG}$ ($X^G = \emptyset \Rightarrow X^G = \emptyset$)

<u>Lemma</u>: $X^{hG} = \left\{ \text{space of sections } \begin{tikzcd} X_{hG} \arrow[d] \\ BG \arrow[u, dashed, bend left] \end{tikzcd} \right\}$

<u>Proof</u>: • $EG \to X \rightsquigarrow$

$$\begin{tikzcd} EG \times EG \arrow[r] & X \times EG \arrow[d] \\ EG \arrow[u, "\mathrm{diag.}"] \arrow[r] & EG \end{tikzcd} \quad /G \rightsquigarrow \quad \begin{tikzcd} & X_{hG} \arrow[d] \\ BG \arrow[ur, dashed] \arrow[r, equal] & BG \end{tikzcd}$$

•
$$\begin{tikzcd} X \times EG \arrow[r] \arrow[d, shift left] & (X \times EG)/G \arrow[d, shift left] \\ EG \arrow[r] \arrow[u, shift left, "f"] & BG \arrow[u, shift left] \end{tikzcd} \qquad EG \xrightarrow{f} X \times EG \to X$$

<u>Prop</u>: If $f: X \to Y$ hG-equiv. then it induces equivalence $f_{hG}: X_{hG} \to Y_{hG}$ and $f^{hG}: X^{hG} \to Y^{hG}$

<u>Proof</u>: •
$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & Y \arrow[d] \\ X_{hG} \arrow[r] \arrow[d] & Y_{hG} \arrow[d] \\ BG \arrow[r, "\simeq"] & BG \end{tikzcd}$$

fibration $\rightsquigarrow$ $X_{hG} \to Y_{hG}$ is a h.equiv.

• f^{hG}-eq: $\begin{tikzcd} X_{hG} \arrow[rr, shift left, "\simeq"] \arrow[dr] & & Y_{hG} \arrow[ll, shift left] \arrow[dl] \\ & BG & \end{tikzcd}$ identifies $\mathrm{Sec}\left(\begin{smallmatrix} X_{hG} \\ \downarrow \\ BG \end{smallmatrix} \right)$ with $\mathrm{Sec}\left(\begin{smallmatrix} Y_{hG} \\ \downarrow \\ BG \end{smallmatrix} \right)$

Prop: $(EG\times -)$ and $\mathrm{map}(EG,-)$ } turns hG-equivalences into G-equivalences

$\mathrm{map}(EG,X)$ & $\mathrm{map}(EG,Y)$ G-equiv.
$\Updownarrow$

In particular X & Y are hG-equiv $\Leftrightarrow$ $X\times EG$ & $Y\times EG$ G-equiv.

Proof: $f: X\to Y$ hG-equiv. $\mapsto$ $\mathrm{map}(EG,X)\xrightarrow{f}\mathrm{map}(EG,Y)$. By Whitehead's thm to show that f is an G-equiv, it suf. $H\leq G$

$$\begin{array}{ccc} \mathrm{map}(EG,X)^H & \xrightarrow{f} & \mathrm{map}(EG,Y)^H \\ \| & & \downarrow \\ X^{hH} & \longrightarrow & Y^{hH} \end{array}$$

is h.eq. ↑ (E^H) is a equiv. by above Prop.

$\Leftrightarrow$ $\begin{array}{c} X_{hG} \\ \downarrow \\ BG \end{array}$ eq. to $\begin{array}{c} Y_{hG} \\ \downarrow \\ BG \end{array}$ as spaces over BG. (fiberwise h.eq)

($\Leftarrow$ defined the same element in $[BG, BAut(X)]$ $(X\simeq Y)$)

Exercise:
$X = \mathrm{hocolim}_{C} F$ is a simplicial space $X_n = \coprod_{x_0\to\cdots\to x_n} F(x_0)$

$$|X| = \Big(\coprod_{x_0\to\cdots\to x_n} F(x_0)\times\Delta^n\Big)\Big/\sim$$

$$\coprod_{x_0\to\cdots\to x_n} F(x_0) \ni (\underbrace{x_0\xrightarrow{\alpha}\cdots\to x_n}_{C},\ \underset{F(x_0)}{y})$$

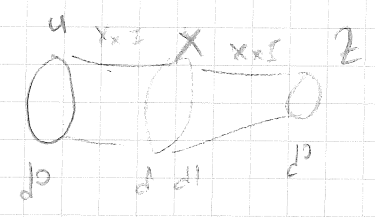
$\downarrow d_i$ $\quad i>0$

$(x_0,\cdots\to x_{i-1}\to x_{i+1}\to\cdots\to x_n,\ y)$

d_0

$(x_1\to\cdots\to x_n,\ F(\alpha)(y))$

Ex: $Y\leftarrow X\to Z$



= double mapping cylinder

$E\mathcal{O}(G) = G/e \downarrow \mathcal{O}(G)$

obj: $\begin{matrix} G/e \\ \downarrow \\ G/H \end{matrix}$ morphisms:
$$\begin{tikzcd} & G/e \arrow[dl] \arrow[dr] & \\ G/H \arrow[rr] & & G/H' \end{tikzcd}$$

$F: \mathcal{O}(G)^{op} \longrightarrow \text{Spaces}$

$C(F) = \operatorname{hocolim}_{E\mathcal{O}G} F = B(\underset{\substack{\shortparallel \\ \text{the identity functor}}}{E^{\infty}}, \mathcal{O}(G)^{op}, \overset{\text{cont.}}{F}) \longrightarrow$

$X = \operatorname{hocolim}_{\mathcal{C}^{op}} F$

simplicial space

$X = \coprod_{x_0 \to \dots \to x_n} E(x_0) \times F(x_n)$

$|X| = (\coprod_{x_0 \to \dots \to x_n} E(x_0) \times F(x_n) \times \Delta^n) / \sim$

Baby case:

$G = \mathbb{Z}/p = \langle t \rangle$

$$\begin{tikzcd} G/G & G/e \arrow[l] \arrow[loop right, "G"] \\ F = Y_0 \arrow[r] & Y_1 \arrow[loop right, "G"] \end{tikzcd}$$

$C(F) =$ homotopy pushout

$$\begin{tikzcd} EG \times Y_0 \arrow[r] \arrow[d] & Y_0 \\ EG \times Y_1 & \end{tikzcd}$$

$C(F)^{\mathbb{Z}/p} = \text{h.p of} \left(\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y_0 \\ \emptyset & \end{tikzcd} \right)$

G-action on EG obj.

$G/e \xrightarrow{g} G/e \xrightarrow{f} G/H \to \dots \to G/H$

$$EG = \begin{matrix} G/e \\ \downarrow 1 \\ G/e \end{matrix} \quad \begin{matrix} G/e \\ \downarrow t \\ G/e \end{matrix} \quad \begin{matrix} G/e \\ \downarrow t^2 \\ G/e \end{matrix}$$

$$\begin{matrix} \downarrow \\ G/e \\ \downarrow \\ G/G \end{matrix}$$