Group cohomology & group actions III — A. Adem, Copenhagen Aug 2011

$G \rightsquigarrow BG \qquad H^*(G;\mathbb{F}_p)$.

$EG \times_G X \qquad$ P.A. Smith theory
localization theorem.

What can this approach say about free actions?

<u>Case:</u> $X \simeq S^n$, with a free G-action.

<u>p^2 condition:</u> If G acts freely on X every abelian subgp is cyclic (i.e. $\mathbb{Z}/p \times \mathbb{Z}/p$ does not occur as a subgp).

$\boxed{\text{homological}}$ condition.

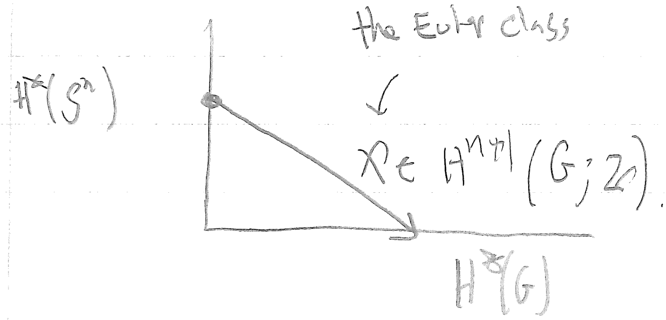
<u>$2p$ condition:</u> (Milnor) If G acts freely on X, then every involution is central.

$\boxed{\text{geometric condition}}$

From Ian's talk: "$2p + p^2 = MTW$"

Thm (MTW) G acts freely on <u>some</u> such X
$\iff$ G satisfies p^2 and $2p$ for all p.

$X = S^n$ $\quad G$ acts freely $\quad EG \times_G X \simeq X/G$ preserving orientation.



The spectral seq deg. to Gysin seq.

$$\cdots \to H^{i+n}(X/G) \to H^i(G) \xrightarrow{\cup x} H^{i+n+1}(G) \to H^{i+n+1}(X/G) \to \cdots$$

So in high dim: $\cup x$ induces iso, since X/G fl. dim.

$\Rightarrow$ Cohomology $H^*(G)$ periodic of period $n+1$. (X is a periodicity generator).

<u>Note</u>: this holds for all subgps of G so they all have periodic cohomology.

But $H^*(\mathbb{Z}/2 \times \mathbb{Z}/2)$ does not have periodic cohom

$\mathbb{Z}/p \times \mathbb{Z}/p \subsetneq G$.

This means Sylow p-subgp cyclic or gen quaternionic. ($p=2$.)

<u>Eg.</u> Q_8:

$$x \in H^4(Q_8; \mathbb{Z}) \cong \mathbb{Z}/8$$

$$H^i(Q_8; \mathbb{Z}) \cong \begin{cases} 0 & i \text{ odd.} \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & i \equiv 2 \ (4) \\ \mathbb{Z}/8 & i \equiv 0 \ (4) \end{cases}$$

V unitary orthogonal G-space

$S(V) \circlearrowleft G$.

(wolf's book)

<u>Thm</u> G will act freely in some $S(V)$ $\Longleftrightarrow$

1) Every subgp of order pq is cyclic.
2) G does not contain $SL_2(\mathbb{F}_p)$ $p>5$ as a subgroup.

What about other manifolds?

M closed manifold with a free G-action preserving orientation.

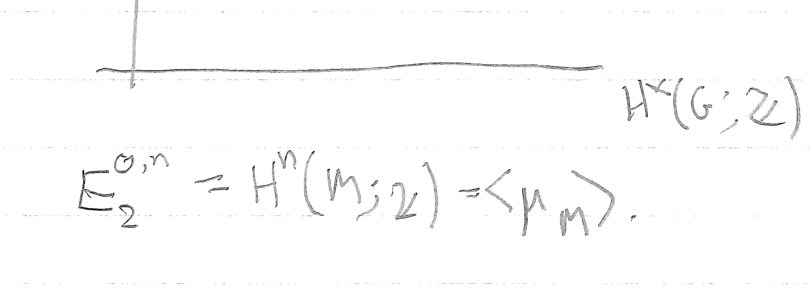
$M \to M/G$ covering space

$|G| \longrightarrow 1$.

$$\mathbb{Z} = H^n(M/G;\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z}) = \mathbb{Z}$$

$$\mu_{M/G} \longmapsto |G|\mu_M.$$

Plug it into the spectral sequence.



$$E_2^{0,n} = H^n(M;\mathbb{Z}) = \langle \mu_M \rangle.$$

$$E_3^{0,n} \hookrightarrow E_2^{0,n} \xrightarrow{d_2} E_2^{2,n-1}$$

$$E_4^{0,n} \hookrightarrow E_3^{0,n} \xrightarrow{d_3} E_3^{3,n-2}$$

$$\vdots$$

$$E_{k+1}^{0,n} \hookrightarrow E_k^{0,n} \xrightarrow{d_k} E_k^{k,n-k+1}$$

$\langle |G| \mu_M \rangle = \operatorname{im} i^* = E_\infty^{0,n} \subset \ldots \qquad \subset E_3^{0,n} \subset E_2^{0,n} = \langle \mu_M \rangle$.

Def: A f.t. ab grp

$$\exp(A) = \min\{n > 0 \mid nA = 0\}.$$

Observe by previous discussion: $|G| \mid \prod_{k=2}^{n+1} E_k^{k,n-k+1}$

Thm (Browder) If G acts freely on a closed manifold M preserving orientation, then

$$|G| \mid \prod_{k=1}^{n} \exp H^{k+1}(G; H^{n-k}(M, \mathbb{Z}))$$

Application:

- $M = pt$. No free action.
- $M = S^n$
  $|G| \mid \exp H^{n+1}(G; \mathbb{Z})$.

$|G| \mid \exp H^{n+1}(G, \mathbb{Z}) \Rightarrow G$ periodic cohomology.

(Since it would hold for $G = \mathbb{Z}/p \times \mathbb{Z}/p$ as well if free action and it doesn't (exponent p not p^2)).

What about products of spheres? Or other spaces?

Corollary: Assume that $G = (\mathbb{Z}/p)^r$ action freely on M + trivially on $H^*(M;\mathbb{Z})$.

then $\hat{H}^i(M;\mathbb{Z}_{(p)}) \neq 0$. for at least r values of i.

~~pf~~:

- p^{r} ~~$\nmid$~~ $p \hat{H}^*((\mathbb{Z}/p)^r;\mathbb{Z}) = 0$ by Künneth.

thm $\Rightarrow$

$$p^r \mid \underbrace{\prod_{k=1}^{n} \exp H^{k+1}((\mathbb{Z}/p)^{r}; H^{n-k}(M;\mathbb{Z}))}$$

So there must be at least m factors in that product.

Corollary (Carlsson)

If $G = (\mathbb{Z}/p)^r$ acts freely on $(S^n)^k$ with trivial action in homology then $r \le k$.

Thm (Adem-Browder)

If $G = (\mathbb{Z}/p)^r$ p odd acts freely on $X = (S^n)^k$

~~then~~ $r \le \dim_{\mathbb{F}_p} H_n(X;\mathbb{F}_p)^G + \frac{1}{p-2}\left[k - \dim H_n(X;\mathbb{F}_p)^G\right]$

$\le k$.

(Conner, late 50s).

Conjecture: If $G = (\mathbb{Z}/p)^r$ acts freely on $S^{n_1} \times \cdots \times S^{n_k}$ then $r \leq k$.

Thm (Hanke 2009) True if $p > 3\dim X$.

X ft. free G-CW-complex, connected.

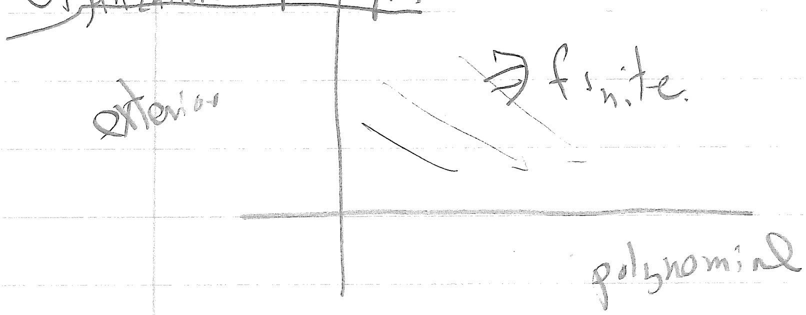
Conjecture (Carlsson) If $(\mathbb{Z}/p)^r$ acts freely on X

then $\dim_{\mathbb{F}_p} H^i(X; \mathbb{F}_p) \geq 2^r$

Special case some conjectures in homological algebra of Horrocks

Note also Carlsson conj $\Rightarrow$ Conner's conj

Organizational principal:



p-rank of G

$$rk_p(G) = \max\{n \mid (\mathbb{Z}/p)^n \subseteq G\}.$$

$$rk(G) = \max_{p \mid |G|} rk_p(G).$$

Homotopy rank of G.

$$h(G) = \min_k \{G \text{ acts freely on a ft. cpx } X \simeq S^{n_1} \times \cdots \times S^{n_k}\}.$$

Conjecture: (Benson - Carlson) ($\forall$ ranks, $\forall$ finite groups)
$r(G) = h(G)$ (May or may not be true).

Swan's result can be restated $rk(G) = 1$.

This is true algebraically (Benson - Carlson) If $r = r(G)$ then $\exists$ a proj $\mathbb{Z}G$-cpx C_* with has the homology of a product of r spheres.

Note: Every f.t. G acts freely on some product of spheres.

What about $X = S^n \times S^m$.

Thm (Conner). $(\mathbb{Z}/p)^3$ does not act freely on $S^n \times S^m$

Some examples from representation theory.
(A - Davis - Ünlü)
$U(n) \supseteq U(n-1) \supseteq U(n-2) \cdots$

$$\begin{pmatrix} A & \begin{matrix} 0 \\ \vdots \end{matrix} \\ \cdots 0 & 1 \end{pmatrix} \longleftrightarrow A$$

$U(n) \quad U(n)/U(1) \quad \cdots \quad U(n)/U(n-1), *$

Def: $G \subset U(n)$ we define the fixity of this rep as the smallest integer f s.t. G acts freely on $U(n)/U(n-f-1)$

$f = 0$ free action on $S^{2n-1} = U(n)/U(n-1)$

$f = n-1$ obvious.

$$U(n-f)/U(n-f-1) \longrightarrow U(n)/U(n-f-1)$$

$$\downarrow$$

$$U(n)/U(n-f)$$

rank 1 isotropy.

p^2-condition

Application: If $G \leq U(n)$ and it has fixity 1, then G acts freely and smoothly on $X = S^{2n-1} \times S^{4n-5}$.

Cor: If $G \leq SU(3)$, ~~then~~ it will act freely and smoothly on $S^5 \times S^7$.

e.g. $SL_3(\mathbb{F}_2)$, A_5, $3A_6$ all act freely on $S^5 \times S^7$.

But: Do not act freely on any $S^n \times S^n$ as they contain A_4.
(result of B Oliver, which has been generalized to

$$\left(SL_2(\mathbb{F}_{2^n}) \sim (\mathbb{Z}/2)^n \rtimes \mathbb{Z}/_{2^n-1}\right.$$

Thm Adem-Davis-Unlu If $p > 3$ prime then a fte-p-grp P acts freely and smoothly on some $S^n \times S^m$ if and only if $r(P) < 3$. (i.e. $(\mathbb{Z}/p)^3 \not\leq P$

pf: $\Rightarrow$ $(\mathbb{Z}/p)^3$ does not act freely on $S^n \times S^m$.

"$\Leftarrow$" $p > 3$ list of rk 2 p-grps
- P acts freely on $S(V) \times S(W)$.
- or $P \leq U(p)$ of fixity one and so P acts freely on $S^{2p-1} \times S^{4p-5}$

Homotopy actions

$$\begin{array}{ccc} S^n & \to & E \\ & & \downarrow \\ & & X \end{array}$$

" periodic.

$$\begin{array}{ccc} U(n-f)/U(n-f-1) & \to & U(n)/U(n-f-1) \\ & & \downarrow \\ & & U(n)/U(n-f). \end{array}$$

isotropy of rank 1.

Eg. $S^n \to X/G$ finite
$$\downarrow$$
$$BG.$$

__Def:__ A space X is said to ~~be~~ have periodic cohomology if there exists a class

$\alpha \in H^*(X;\mathbb{Z})$ and an integer $d \geq 0$.

s.t. $\forall$ coeff M

$$\cup\alpha : H^n(X;M) \to H^{n+|\alpha|}(X;M)$$

is an isomorphism $\forall\ n \geq d$.

__Thm__ (Adem-Smith) A connected CW-cpx X has periodic cohomology if and only if there exists an orientable spherical fib.

$$\begin{array}{ccc} S^n & \to & E \\ & & \downarrow \\ & & X \end{array}$$

s.t. E has the homotopy type of a finite dim cpx

G has periodic cohomology $\iff$ its abelian subgps are cyclic

$\boxed{\text{higher Krull dimensions}} \leftarrow \boxed{\text{Krull dim 1}}$

Applications:

1) M is a simply connected compact manifold with a G-action having isotropy subgroups all rank equal to one.
Then there exists a finite cpx X w. a free Gaction and
$$X \simeq M \times S^N \text{ for some } N.$$
to get an action on $S^n \times S^m$ it suffices to get an action on S^n w. periodic isotropy.

$$\begin{array}{ccc} S^m & \to & E \\ & & \downarrow \circlearrowleft G \\ & & S^n \end{array}$$

rank 1.

$G = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(p)$
acts on a sphere then the isotropy has rank 2.

<u>Possible goal</u>: Show that $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(\mathbb{F}_p)$ does <u>not</u> acts freely on a product of 2 spheres.

If P p-group rank$(P) = 3$ then it acts freely on $S^n \times S^m \times S^k$ (M. Klaus).

Thm (Wall, A-S) A discrete gp Γ acts freely and properly on some

$S^n \times \mathbb{R}^k \iff \Gamma$ is a countable gp with periodic cohomology.