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Group cohomology & group actions II

diag G-action.

Borel Construction: $X_{hG} = (X \times EG)/G = X \times_G EG$.

① Have fibration

$$\begin{array}{ccc} X & \to & EG \times_G X \\ & & \downarrow \\ & & EG/G. \end{array}$$

② Have map

$$\begin{array}{c} EG \times_G X \\ \downarrow \\ X/G \end{array}$$

Not a fibration in general.

Fiber over σ is BG_σ. Varies with σ.

In the special case where action on X is free
$EG \times_G X \to X/G$ h.e.

<u>Note</u>: over $\mathbb{Q}$: $EG \times_G X \underset{\mathbb{Q}}{\simeq} X/G$. (Vietoris Begle).
(same homology $\mathbb{Q}$)

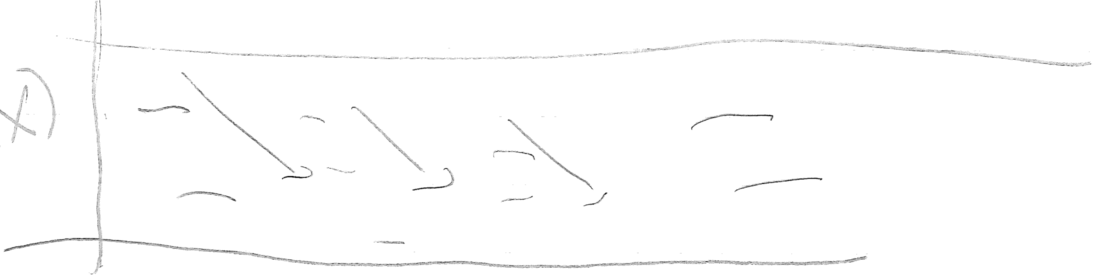
Borel Equivariant cohomology $H^*_{hG}(X) = H^*(EG \times_G X)$

Serre spectral sequence X

$$E_2^{p,q} = H^p(G; H^q(X; R)) \Rightarrow H^{p+q}(EG \times_G X; R).$$

X finite dimensional

$H^*(X)$



$H^*(BG)$ inf. dim.

Thm (P.A. Smith) If X is a finite dimensional space with the mod p homology of a pt. and if a finite p-grp P acts on X, then $X^P \neq \phi$ and has the mod p homology of a pt.

Technical fact: X f.d. P-space, $P = \mathbb{Z}/p$ then

$$H^*(EP \times_P X, EP \times_P X^P; \mathbb{F}_p) = H^*(X/P, X^P/P)$$
$$= H^*_P(X, X^P)$$

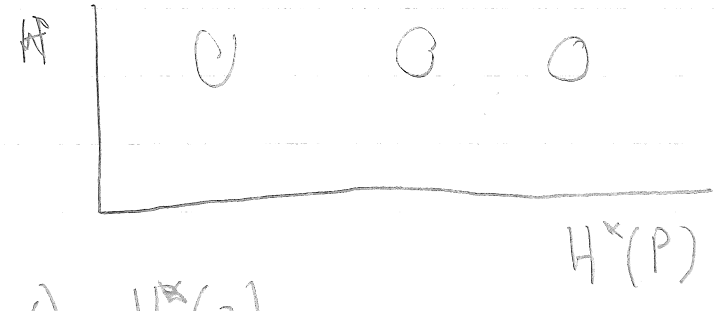
finite dimensional

Pf: see Bredon: compact transformation groups.

<u>Conclusion:</u> $H^*(EP \times_P X; \mathbb{F}_p) \to H^*(EP \times_P X^P; \mathbb{F}_p)$ is an isomorphism for $* > \dim X$.

If $X \underset{p}{\sim} *$

$$E_2^{p,q} = H^p(P; H^q(X)) = H^p(P)$$



$$H^*(EP \times_P X) \cong H^*(P)$$
$$\cong H^*(EP \times_P X^P) \qquad * \text{ big.}$$
$$\cong$$
$$H^*(P) \otimes H^*(X^P)$$

RHS LHS

~~Now~~ $H_P^*(X) = H^*(P)$ $H^*(X^P) \otimes H^*(X)$

Since these agree for $\underline{n \text{ large}}$

we get $H^*(X^P) \cong H^*(X \times pt.$

Likewise

<u>Thm</u> (P.A. Smith) $X \simeq_p S^n$ and $P = \mathbb{Z}/p$ acts non-freely
~~then~~ $X^P \simeq_p S^k$ for some k.

$1 \to H \to G \to K \to 1.$

Can we compute $H^*(G)$ from $H^*(K)$ and $H^*(H)$

Lyndon Hochschild Serre s.s.

$\rightsquigarrow$ fibration $BH \to BG \to BK$.

So get

$$E_2^{p,q} = H^p(K; H^q(H)) \Rightarrow H^{p+q}(G).$$

<u>Examples:</u> $G = A_4$

$$1 \to \mathbb{Z}/_2 \times \mathbb{Z}/_2 \to A_4 \to \mathbb{Z}/_3 \to 1.$$

Only $\mathbb{F}_p$ cohomology for $p = 2, 3$.

$$H^*(A_4; \mathbb{F}_3) = H^p(\mathbb{Z}/_3; \underbrace{H^q(\mathbb{Z}/_2 \times \mathbb{Z}/_2; \mathbb{F}_3)}_{\substack{0 \text{ unless } q = 0 \\ \text{when } q = 0}})$$

$$= H^*(\mathbb{Z}/_3; \mathbb{F}_3) \cong H^*(A_4; \mathbb{F}_3)$$

$H^*(A_4;\mathbb{F}_2)$

$$E_2^{p,q} = H^p(\mathbb{Z}/3; H^*(\mathbb{Z}/2\times\mathbb{Z}/2;\mathbb{F}_2)) \cong \begin{cases} 0 & p>0 \\ \mathbb{F}_2[x_1,x_2]^{\mathbb{Z}/3} & p=0. \end{cases}$$

$$= \mathbb{F}_2[u_2, v_3, w_3]/u_2^2+v_3w_3+v_3^2+w_3^2$$

(see Adem–Milgram).

<u>Exercise</u>: Show: $H^*(A_5;\mathbb{F}_2) \cong H^*(A_4;\mathbb{F}_2)$.

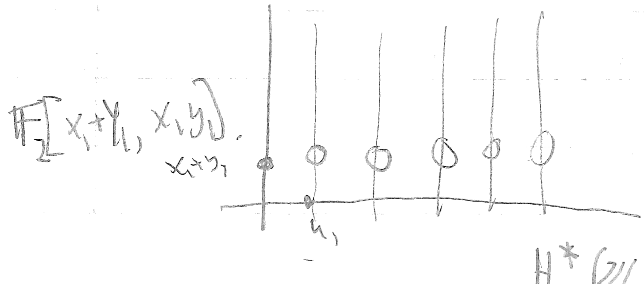
$H^*(D_8;\mathbb{F}_2)$ $\qquad D_8 = \mathbb{Z}/2 \wr \mathbb{Z}/2$.

$$\underbrace{\mathbb{Z}/2\times\mathbb{Z}/2}_{\circlearrowright\, T} \rtimes \mathbb{Z}/2$$

$$\mathbb{Z}/2\times\mathbb{Z}/2 \to D_8 \to \mathbb{Z}/2$$

$$H^p(\mathbb{Z}/2; H^q(\mathbb{Z}/2\times\mathbb{Z}/2))$$

$$H^*(\mathbb{Z}/2\times\mathbb{Z}/2)^{\mathbb{Z}/2} = \mathbb{F}_2[x_1,x_1]^{\mathbb{Z}/2} \cong \mathbb{F}_2[\underset{\deg 1}{x_1+y_1}, \underset{\deg 2}{x_1y_1}].$$



$H^*(\mathbb{Z}/2) \cong \mathbb{F}_2[u_1]$.

$u_1(x_1+y_1) = 0$ because $x_1+y_1 = x_1 + Tx_1$

$$E_2^{*,*} = \mathbb{F}_2[x_1+y_1, x_1y_1, u_1] / u_1(x_1+y_1) = 0$$

No differentials ~~since sections~~. (look at degrees, use section)

$= E_\infty^{*,*}$ In fact

$H^*(D_8; \mathbb{F}_2) = H^*(\mathbb{Z}/2; H^*(\mathbb{Z}/2 \times \mathbb{Z}/2))$. $\square$

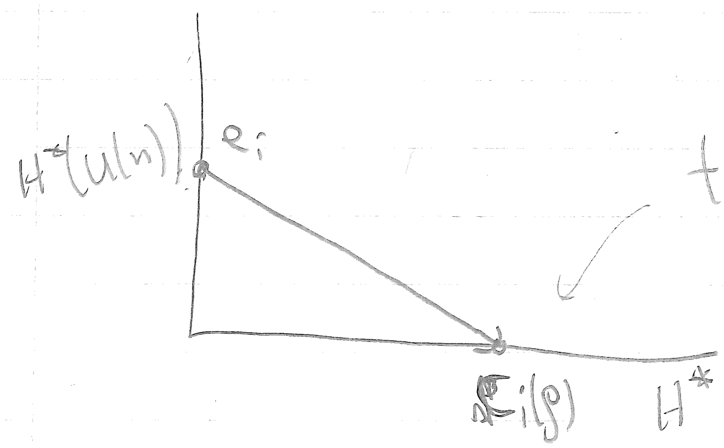
$G \hookrightarrow U(n)$ unitary representation.

G acts by translations on $U(n)$

Free action.

Have fibration sequence

$$\begin{array}{ccccc} U(n) & \to & U(n)/G & \to & BG \\ & & & & \downarrow \\ & & & & BU(n) \end{array}$$



transgresses by result of Borel.

$c_i(\rho)$ = Chern classes of the representation

$H^*(U(n); \mathbb{Z}) = \Lambda(e_1, e_3, \ldots e_{2n-1})$.

$H^*(BG; H^*(U(n))) = H^*(BG) \otimes H^*(U(n))$.

$\swarrow$ f.t. cohomology!

Converges to $H^*(U(n)/G)$

Have map

controlled by $H^*(U(n)/G)$

$$H^*(BU(n)) \longrightarrow H^*(BG).$$

$$H^*(BU(n);\mathbb{F}_p) \cong \mathbb{F}_2[c_1, \ldots, c_n].$$

polynomial ring.

<u>Thm</u> (Venkov). $H^*(BG;\mathbb{F}_p)$ is finitely generated as a module over $H^*(BU(n);\mathbb{F}_p)$.

In particular $H^*(BG;\mathbb{F}_p)$ noetherian.

$$P_G(t) = \sum_{i=0}^{\infty} \dim_{\mathbb{F}_p} H^i(G;\mathbb{F}_p)\, t^i$$

poincaré series.

$$P_G(t) = \frac{r(t)}{\prod_{i=1}^{n}(1-t^{2i})}.$$

What is the order of pole at $t=1$?

<u>Ex</u>: • $P_{\mathbb{Z}/2}(t) = \dfrac{1}{1-t}$

• $P_{(\mathbb{Z}/2)^n}(t) = \dfrac{1}{(1-t)^n}$ $\qquad P_{D_8}(t) = \dfrac{1}{(1-t)^2}$

<u>Thm</u>: (Quillen, ~~Quillen~~ conj of Atiyah–Swan).
order $= \max\{m \mid (\mathbb{Z}/p)^m \subseteq G\}$.

Ex: $G = Q_8 \hookrightarrow SU(2) = S^3$.

$H^*(BSU(2)) = \mathbb{F}_2[u_4]$.

Algebraic extension

$$0 \to p^*(u_4^4) \to H^*(Q_8) \to H^*(SU(2)/Q_8) \to 0.$$

$$H^*(Q_8) = \mathbb{F}_2[x_1, y_1] \Big/ \begin{matrix} x_1^2 + x_1y_1 + y_1^2 \\ x_1^2y_1 + x_1y_1^2 \end{matrix}$$

$$P_{Q_8}(t) = \frac{1 + 2t + 2t^2 + t^3}{1 - t^4}.$$

$$A_p(G) = \left\{ \begin{matrix} \text{poset of nontrivial elementary abelian } p\text{-groups} \\ \text{in } G. \end{matrix} \right\}$$

$(\mathbb{Z}/p)^n$ "p-tori".

$E \in A_p(G)$ $\qquad H^*(G) \xrightarrow{\text{res}} H^*(E)$.

Thm (Quillen)

$$H^*(G) \xrightarrow{Q} \lim_{E \in A_p(G)} H^*(E).$$

is an F-isomorphism.

i.e. • $\ker Q$ is nilpotent.

• Given $x \in \lim\limits_{E \in A_p(G)} H^*(E)$.

$\exists N$ s.t. $\quad x^{p^N} \in \operatorname{Im} Q$.

i.e. "up to nilpotence" mod p cohomology is detected on elementary abelian subgps.

<u>Def:</u> A collection of subgps $H_1, \ldots, H_n$ is said to detect if $H^*(G;\mathbb{F}_p)$ if

$$\bigoplus_i \mathrm{res}^G_{H_i} : H^*(G;\mathbb{F}_p) \to \bigoplus_i H^*(H_i) \text{ is } \underline{\text{injective}}.$$

$p=2$

<u>Thm</u> If $G = \Sigma_n$, then its mod 2 cohomology is detected on elementary abelian subgps.

Fact. $$H^*(\Sigma_n) = \lim_{E \in A_2(\Sigma_n)} H^*(E).$$

1960' Nakaoka, Steenrod, Milgram, Feshbach $H^*(\Sigma_\infty)$.
Sinha - Salvatore = Giusti 2011:

$$H^*(\Sigma_\infty) \twoheadrightarrow H^*(\Sigma_n)$$

<u>Question:</u> What about <u>simple</u> gps.

$L_3(2)$ $\quad |A_2(G)|/G$ $\quad \Sigma_4 \underset{D_8}{\circ\!\!-\!\!-\!\!-\!\!-\!\!\circ} \Sigma_4.$

$$H^*(L_3(2)) = \mathbb{F}_2[u_2, v_3, w_3]/v_3w_3.$$

A_6: $$H^*(\Sigma_6)/(\sigma_1). \cong H^*(A_6).$$

$$H^*(A_6) \cong H^*(L_3(2))$$

however no nontrivial maps.

$$\begin{tikzcd}
A_6 & & L_3(2). \\
& \Sigma_4 *_{D_8} \Sigma_4 \arrow[ul, "\cong"] \arrow[ur] &
\end{tikzcd}$$