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Cohomology & Action of finite gps. I

G fi. gp. (but also applies to discrete gps or cpt Lie gps).

Cohomology of G: $P_* \to \mathbb{Z}$ projective resolution.
← triv action.

$$\cdots P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to \mathbb{Z} \to 0$$

P_i projective $\mathbb{Z}G$-mod.
$\operatorname{im} \partial_i = \ker \partial_{i-1}$ $i \geq 2$.
$P_0/\partial_1 \cong \mathbb{Z}$

M $\mathbb{Z}G$-module
$\operatorname{Hom}_G(P_*, M)$

__Def:__ $H^i(G; M) := H^i(\operatorname{Hom}_G(P_*, M))$. $i = 0, 1, \ldots$

$$=: \operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M).$$

__Prop:__ This does __not__ depend on choice of resolution.

In particular $H^i(G; \mathbb{Z})$ homological invariant of $\mathbb{Z}G$.

__Example:__ $G = \mathbb{Z}/N = \langle t \mid t^N = 1 \rangle$.

Have two maps: $\mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G$

$x \longmapsto (1-t)x$

$\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G$ Norm map.

$x \longmapsto (1 + t + \cdots + t^{N-1})x$

$$\cdots \to \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$
$$g \longmapsto 1.$$

Exercise: This defines a free resolution.

This defines a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/k]$.

So

$$H^i(\mathbb{Z}/k; M) = H(0 \to M \xrightarrow{1-t} M \xrightarrow{N} M \xrightarrow{1-t} M \xrightarrow{N} \cdots)$$

so $$H^i(\mathbb{Z}/k; M) = \begin{cases} M^G & i = 0 \\ M^G/NM & i \text{ even } i > 0 \\ \ker N/(1-t)M & i \text{ odd.} \end{cases}$$

In particular $$H^i(\mathbb{Z}/k; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/k & i \text{ even } i > 0 \\ 0 & i \text{ odd.} \end{cases}$$

~~In particular: $H^i(\mathbb{Z}/k; \mathbb{Z}) =$~~

In particular $H^i(\mathbb{Z}/k; M) \cong H^{i+2}(\mathbb{Z}/k; M)$
2-fold periodicity.

What about general construction?

X_n = free $\mathbb{Z}G$-module w basis G^n
$[x_1 | \ldots | x_n]$ $n > 0$ $[\;]$ $n=0$.

$$X_* \to \mathbb{Z} \qquad \varepsilon([\,]) = 1$$

$$\begin{aligned} d_n([x_1|\dots|x_n]) &= x_1[x_2|\dots|x_n] \\ &+ \sum_{i=1}^{n-1} (-1)^i [x_1|\dots|x_i x_{i+1}|\dots|x_n] \\ &+ (-1)^n [x_1|\dots|x_{n-1}]. \end{aligned}$$

Prop: This defines a free resolution of $\mathbb{Z}$.

Pf: Exercise.

So $H^n(G;M) = H^n(\mathrm{Hom}_G(C_*(EG), M))$

~~Bar~~ This is called the bar resolution.

Functorial in G.

Back to topology:

Def: If G acts on X we say that the action is free if $G_x = \{g \in G \mid gx = x\} = 1 \quad \forall x \in X$.

EG denotes a free G-space, which is contractible.

Prop: $C_*(EG) \to \mathbb{Z} \to 0$. free resolution.

~~Pf:~~ Exercise

$$\begin{array}{l} EG \\ \downarrow \\ BG = EG/G \end{array}$$

space of orbits

$$H^*(BG;M) := H^*(\mathrm{Hom}_G(C_*(EG), M))$$

Hence $H^n(G;M) = H^n(BG;M)$!

What is EG ?

There is a model for EG.

Milnor's model: (using joins)

Formal sums $\sum t_i g_i$ s.t. $t_i = 0$
except for finitely many i. $g_i \in G$.
$\sum t_i = 1$, $t_i \geq 1$.
$0g = 0g'$.

Topology: Smallest topology s.t.
$t_i : EG \to [0,1]$.
$g_i : t_i^{-1}(0,1] \to G$

$(\sum t_i g_i) g = \sum t_i g_i \cdot g$ G-action which is free

Ex: $EG \downarrow BG = EG/G$ principal G-bundle locally trivial

(Milnor, Steenrod?)

Thm This is a universal principal G-bundle in the sense that any other (nice) principal G-bundle is pulled back from this one. i.e.

$\{\text{principal G-bundles}/X\} \leftrightarrow [X, BG]$

BG is the classifying space of G.

EG for G discrete is contractible

$$EG = \operatorname{colim}_n (\underbrace{G * G * \cdots * G}_{n\text{-fold join}})$$

So we have functor

$$G \longmapsto BG.$$

grp — classifying space.

is a functor

$$\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$$ are unique up to homotopy.

<u>Example:</u> 1) $G = \mathbb{Z}/2$

$$S^n \xrightarrow{A} S^n$$

$Ax = -x$ antipodal action

$$S^\infty = \operatorname{colim}_n S^n.$$

We get universal principal $\mathbb{Z}/2$-bundle.

$$\begin{array}{c} S^\infty \circlearrowleft \mathbb{Z}/2 \\ \downarrow \\ \mathbb{R}P^\infty \end{array}$$

$$H^*(G; \underset{\text{ring}}{R}) := H^*(BG; R).$$

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[x_1]/x_1^{n+1}$$

$$H^*(\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x_1].$$

2) p odd.

$\mathbb{Z}/p$ acts freely on S^{2n+1} quotient $L_p^{2n+1} = S^{2n+1}/\mathbb{Z}/p$

Lens space

$$\begin{array}{c} S^\infty \\ \downarrow \\ L_p^\infty = B\mathbb{Z}/p \end{array}$$ Lens space.

$$H^*(L_p^{2n+1}; \mathbb{F}_p) \simeq \Lambda(x_1) \otimes \mathbb{F}_p[x_2]/x_2^{n+1}$$

$$H^*(\mathbb{Z}/p; \mathbb{F}_p) \simeq \Lambda(x_1) \otimes \mathbb{F}_p[y_2].$$

3) $E = (\mathbb{Z}/p)^\ell$ elementary ab p-grp.

$$H^*(E; \mathbb{F}_p) \simeq \begin{cases} \mathbb{F}_2[x_{1,1}, \ldots, x_{1,n}] & p=2 \\ & |x_{1,i}| = 1. \\ \Lambda(x_{1,1}, \ldots x_{1,n}) \otimes \mathbb{F}_p[y_{2,1}, \ldots y_{2,n}] & p \text{ odd} \end{cases}$$

$|x_{1,i}| = 1$

$|y_{2,i}| = 2.$

4) $G = S^1$ $$\begin{array}{c} S^{2n+1} \\ \downarrow \pi \\ \mathbb{C}P^n \end{array}$$ principal S^1-bundle.

$BS^1 = \mathbb{C}P^\infty$

$ES^1 = S^\infty$

5) $U(n)$ $n\times n$ - unitary matrices.

$BU(n) = ??$

$BU(n) =$ Grassmanian of n-planes in $\mathbb{C}^\infty$.

$$\begin{array}{c} EU(n) = \text{stiefel manifolds.} \\ \downarrow \; /U(n) \\ BU(n) \end{array}$$

$G \hookrightarrow U(n)$ any f.t. gp occurs as subgp in $U(n)$.

So have $BG \to BU(n)$.

$H^*(BU(n); \mathbb{F}_p) \to H^*(BG; \mathbb{F}_p)$

$\shortparallel$

$\mathbb{F}_p[c_1, \ldots, c_n]$ $|c_i| = 2i$. Chern classes

$\circlearrowright \Sigma_n$.

$(BS^1)^n \longrightarrow BU(n)$

$H^*(BU(n)) \xrightarrow{\cong} \mathbb{F}_p[z_1, \ldots, z_n]^{\Sigma_n}$

$|z_i| = 2$.

Subgps / groups in finite group cohomology.

$H \subset G$ M $\mathbb{Z}G$-module

$C_*(EH) \to C_*(EG)$

EG has a free H-action and is contractible, so works as EH.

Get map $EG/H \to EG/G$

$$\begin{array}{ccc} C_*(EG/H) & \to & C_*(EG/G) \\ \wr & & \\ C_*(BH) & \to & C_*(BG) \end{array}$$

$$H^*(G;M) \to H^*(H; M_{|H})$$

Other way around?

$$\mathrm{Hom}(C_*(EG), \mathbb{Z})^G \qquad\qquad M^G$$
$$\downarrow \quad \uparrow ? \qquad\qquad\qquad \downarrow$$
$$\mathrm{Hom}(C_*(EG), \mathbb{Z})^H \qquad\qquad M^H.$$

Transfer map: $M^H \to M^G$

$$x \longmapsto \sum_{[g]\in G/H} gx$$

Note: well def and lands in M^G.

$$\mathrm{Hom}(C_*(EG), M)^H \longrightarrow \mathrm{Hom}(C_*(EG), M)^G$$

$$f \longmapsto \sum_{[g] \in G/H} gf.$$

Gives map

$$H^*(H; M\downarrow_H) \longrightarrow H^*(G; M).$$

transfer or corestriction map.

Basic properties of res and tr:

- $H \subset K \subset G$ $\qquad \mathrm{res}^K_H \circ \mathrm{res}^G_K = \mathrm{res}^G_H.$
- res compatible w. product, but tr is not.
- $\mathrm{tr}^G_H \, \mathrm{res}^G_H(x) = [G:H]\, x.$
- $G = \coprod_{i=1}^m H g_i K \qquad H \hookrightarrow G \hookleftarrow K$
- $\mathrm{res}^G_H \mathrm{tr}^G_K = \sum_{i=1}^m \mathrm{tr}^H_{H \cap g_i^{-1} K g_i} \circ \mathrm{res}^{g_i^{-1} K g_i}_{H \cap g_i^{-1} K g_i} c_{g_i}$

c_{g_i} map induced by conj by g_i

Consequences:

1) $|G| \, H^*(G;M) = 0 \quad * > 0.$

pf $\quad \underbrace{H^*(G;M) \to H^*(e;M) \to H^*(G;M)}_{|G|}$

0 in $* > 0$.

2) $S \leq G$ Sylow p-subgp.

$[G:S]$ prime to p.

$$H^*(G;M) \hookrightarrow H^*(S;M)$$

Using double coset formula:

$$H^*(G;M) = \lim_{\substack{P \ p\text{-subgps} \\ \leq G}} H^*(P;M). \qquad =: L$$

L consists of sequences of cohomology classes $x_P \in H^*(P;\mathbb{F}_p)$ s.t.

1) if $P' \subset P$ then

$x_{P'} = \mathrm{res}^{P}_{P'} \, x_P.$

2) if $P' = gPg^{-1}$ then $x_{P'} = c_g \, x_P$

Thm (Swan) If G has abelian Sylow p-subgp S, then

$$H^*(G;\mathbb{F}_p) \cong H^*(N_G(S);\mathbb{F}_p).$$