Pre-exercises on fusion systems, etc.

July 14, 2011

For a category $\mathcal{C}$ with objects a and b, we write $\mathcal{C}(a,b)$ to denote $\mathrm{Hom}_{\mathcal{C}}(a,b)$ and $\mathcal{C}(a)$ for $\mathrm{Aut}_{\mathcal{C}}(a)$.

Let S be a p-group. Recall that a *fusion system on* $\mathcal{F}$ is a category whose objects are the subgroups of S and whose morphisms are some collection of injective group homomorphisms between the subgroups such that (1) every map induced by conjugation by some element of S is a morphism of $\mathcal{F}$ and (2) every morphism factors as an isomorphism (in $\mathcal{F}$) followed by an an inclusion.

Recall that a subgroup $P \leq S$ is called *fully normalized (in $\mathcal{F}$)* (resp., *fully centralized*) if $|N_S(P)| \geq |N_S(P')|$ (resp., $|C_S(P)| \geq |C_S(P')|$) for all P' $\mathcal{F}$-isomorphic to P. Then we say that $\mathcal{F}$ is *saturated* if

(I) Every fully normalized subgroup P is fully centralized, and moreover $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathcal{F}(P))$.

(II) If Q is fully normalized and $\varphi \in \mathcal{F}(P,Q)$ is an isomorphism, then there exists an extension of φ in $\mathcal{F}(N_\varphi, S)$, where
$$N_\varphi := \left\{ n \in N_S(P) \middle| \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(Q) \right\}.$$

Exercise 1. Suppose that G is a finite group with $S \in \mathrm{Syl}_p(G)$. Let $\mathcal{F}_S(G) = \mathcal{F}$ denote the category whose objects are the subgroups of S and with $\mathcal{F}(P,Q) = \mathrm{Hom}_G(P,Q)$ is the set of morphisms induced by conjugation by some element of G. Show that $\mathcal{F}$ is a saturated fusion system.

Exercise 2. Find a finite group G with a non-Sylow p-subgroup P such that $\mathcal{F}_P(G)$ is saturated.

Given a saturated fusion system $\mathcal{F}$ on S, we say that $P \leq S$ is *$\mathcal{F}$-centric* if for all P' $\mathcal{F}$-isomorphic to P, we have $C_S(P') = Z(P')$. We also define the (minimal) centric transporter system on S to be the category $\mathcal{T} = \mathcal{T}_S^c(S)$ whose objects are the $\mathcal{F}$-centric subgroups of S and where $\mathcal{T}(P,Q) = N_S(P,Q)$ is the set of elements of S that conjugate P into Q.

A *centric linking system on* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S together with functors
$$\mathcal{T}_S^c(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c$$
where $\mathcal{F}^c$ is the full subcategory of $\mathcal{F}$ with objects the $\mathcal{F}$-centric subgroups of S. We require that the following axioms are satisfied:

(A) Both δ and π are the identity on objects, while on morphisms δ is injective and π is surjective. Moreover, $Z(P)$ acts freely on $\mathcal{L}(P,Q)$ (i.e., no nonidentity element of $Z(P)$ stabilizes any element of $\mathcal{L}(P,Q)$) via δ and right composition, and $\pi : \mathcal{L}(P,Q) \to \mathcal{F}(P,Q)$ is the orbit map of this action (i.e., so that $\pi(\mathfrak{g}) = \pi(\mathfrak{h})$ if and only if there is some $z \in Z(P)$ such that $\mathfrak{h} = \mathfrak{g} \cdot z$).

(B) The composite $\pi \circ \delta$ sends any $n \in N_S(P,Q)$ to $c_n \in \mathcal{F}(P,Q)$.

(C) For any $\mathfrak{g} \in \mathcal{L}(P,Q)$ and $a \in P$, the following diagram commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P(a) & \xrightarrow{\mathfrak{g}} & Q \\ {\scriptstyle \delta_P}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(\mathfrak{g})(a))} \\ P & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

Exercise 3. Let G be a finite group with $S \in \mathrm{Syl}_p(G)$, and let $\mathcal{F} = \mathcal{F}_S(G)$ be the induced saturated fusion system on S.

(a) Show that a subgroup $P \leq S$ is $\mathcal{F}$-centric if and only if P is p-centric in G, i.e., iff $Z(P) \in \mathrm{Syl}_p(C_G(P))$.

(b) Define a category $\mathcal{L} = \mathcal{L}_S^c(G)$ whose objects are the $\mathcal{F}$-centric subgroups of S and whose morphisms are defined by $\mathcal{L}(P, Q) = N_G(P, Q)/O^p(C_G(P))$. Show that $\mathcal{L}$ naturally has the structure of a linking system on $\mathcal{F}$.