Exercises F-modules

(1) In (1.2) we have shown that if a best offender acts nontrivially on a submodule, it induces an offender on this submodule too. Now prove that a best offender induces a best offender. Precisely:

Let V a faithful $\mathbb{F}_pG$-module, $A \in \mathcal{P}(G,V)$ and W be a submodule with $1 \neq [A,W] \leq W$. Then $AC_G(W)/C_G(W) \in \mathcal{P}(N_G(W)/C_G(W), W)$.

(2) There is something like a maximal best offender, if we drop the assumption that offender have to be abelian. So show:

Let V be a faithful $\mathbb{F}_p$-module for G. Let A, B be p-groups with $|A||C_V(A)| = |B||C_V(B)|$ maximal. If $\langle A, B\rangle$ is a p-group, then $\langle A, B\rangle = AB$ and $|C_V(AB)||AB| = |A||C_V(A)|$.

(3) Let A be a faithful F-module offender on V, and $V = V_1 \times V_2$ with $1 \neq [V_i, A] \leq V_i$, $i = 1, 2$. Suppose that $|V_i : C_{V_i}(A)| \geq |A/C_A(V_i)|$ (in particular A does not induce an over offender on any V_i), then $A = C_A(V_1)C_A(V_2)$ and $C_A(V_i)$ is an offender on V_{3-i}, $i = 1, 2$.

(4) Let $G = \mathrm{Sym}(5)$ and V be an F-module over $\mathbb{F}_p$ with $C_V(G) = 1$. Show

(i) If $p = 2$, then $|V| = 16$.

(ii) If $p \geq 5$, there is no F-module

(iii) What happens with $p = 3$?

Don't use any outside information about irreducible modules for G, not the classification theorem for F-modules from the lecture nor results about quadratic modules, which were not present in the first lecture.

(5)* Show that $\mathrm{Alt}(6)$ does not possess an F-module over $\mathbb{F}_3$. (Don't not use results about quadratic action from the second lecture.)