Exercises on centric obstruction theory

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1 Definitions

Exercise 1. Let G be a finite group and $S \in \mathrm{Syl}_p(G)$. Recall that $\mathcal{O}_S(G)$ is the category with objects the subgroups of S and with morphisms $\mathcal{P}_S(G)(P,Q) := Q\backslash N_G(P,Q)$. Show that $\mathcal{O}_S(G)$ is equivalent to the category of transitive G-sets whose stabilizers are p-groups and G-maps.

Recall the following from lecture:

Theorem 1.1 (Jackowski-McClure)**.** *If M is a $\mathbb{Z}_{(p)}$-module and $F : \mathcal{O}_S(G)^{op} \to \mathbb{Z}_{(p)}$-mod is the functor $P \mapsto M^P$, then*

$$H^n(\mathcal{O}_S(G); F) = 0$$

for all $n \geq 1$.

Definition 1.2. For Γ a finite group, $\Sigma \in \mathrm{Syl}_p(\Gamma)$, and M a $\mathbb{Z}_{(p)}\Gamma$-module, let $F_M : \mathcal{O}_\Sigma(\Gamma)^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-mod be the functor that vanishes off the trivial subgroup and sends $\{1\}$ to M. Set

$$\Lambda^*(\Gamma; M) := H^*(\mathcal{O}_\Sigma(\Gamma); F_M).$$

Theorem 1.3 (Jackowski-McClure-Oliver)**.** *If $\mathcal{O}$ is either $\mathcal{O}_S(G)$ or $\mathcal{O}(\mathcal{F}^c)$ and $F : \mathcal{O}^{op} \to \mathbb{Z}_{(p)}$-mod is a functor that vanishes off of the isomorphism class of $P \leq S$, then for all $n \geq 0$,*

$$H^n(\mathcal{O}; F) \cong \Lambda^n(\mathrm{Aut}_{\mathcal{O}}(P); F(P)).$$

2 Basic properties

Exercise 2. Show that if $p \nmid |\Gamma|$, then

$$\Lambda^n(\Gamma; M) \cong \begin{cases} M^\Gamma & n = 0 \\ 0 & n \geq 0 \end{cases}.$$

Hint: Does this formula look familiar?

Exercise 3. Show that if $p \big| |\Gamma|$, then $\Lambda^0(\Gamma; M) = 0$.

Exercise 4. Show that if $|\Sigma| = p^n$, then $\Lambda^m(\Gamma; M) = 0$ for all $m > n$.
Hint: Use Jackowski-McClure. Also induction.

Exercise 5. Show that if $|\Sigma| = p$, then $\Lambda^1(\Gamma; M) \cong M^{N_\Gamma(\Sigma)}/M^\Gamma$.

Exercise 6. Let $G = A_6$, $\mathcal{F} = \mathcal{F}_S(G)$, and $\mathcal{Z}_G : \mathcal{O}(\mathcal{F}^c) \to \mathbb{Z}_{(p)}$-mod be the obstruction functor $P \mapsto Z(P)$. Compute the higher limits of $\mathcal{Z}_G$.