A characterization of the 2-fusion system of $L_4(q)$

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Fusion systems and p-local group theory
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Overview and notation

1. The $L_2(q)$, A_7 problem and preliminaries
2. Simple 2-fusion systems
3. An $L_2(q)$ standard form problem in fusion systems
4. Thompson transfer lemma for fusion systems

- All fusion systems will be saturated.
- $O_p(\mathcal{F})$ - the largest normal p-subgroup of $\mathcal{F}$
- $Z(\mathcal{F})$ - the center of $\mathcal{F}$, a normal p-subgroup of $\mathcal{F}$
- $O^p(\mathcal{F})$ on the hyperfocal subgroup $\mathfrak{hyp}(\mathcal{F})$
- $O^{p'}(\mathcal{F})$ on S itself
- The focal subgroup $\mathfrak{foc}(\mathcal{F}) = [S, \mathcal{F}]$
- $L_m(q) = PSL(m, q)$, $U_m(q) = PSU(m, q)$, $S_m(q)$ and $\Omega_m(q)$ the simple symplectic, orthogonal groups.

A "standard form" problem for $L_2(q^2)$, A_7

Problem

Let G be a finite group with an involution $x \in G$ such that $C := C_G(x)$ has a simple normal subgroup K isomorphic to $L_2(q^2)$ (q odd) or A_7 and such that $C_C(K)$ is a cyclic 2-group. Determine G.

- Originally one of the last problems to be completed in CFSG of component type; very difficult partly because of accidental isomorphisms.
- One example:
$$L_2(9) \cong A_6 \cong Sp_4(2)' \cong \Omega_5(2)$$
leading to pesky almost simple groups
$$L_5(2)\langle\gamma\rangle, U_5(2)\langle\gamma\rangle, \text{ and } S_4(4)\langle\varphi\rangle.$$
- One also has to identify S_8, S_9, HS and $L_4(q)$.
- Under weaker hypotheses, $S_4(q^2)$, A_{10}, A_{11}, M_{12}, J_1, J_2, $\mathrm{Aut}(He)$ ...show up.

Normal subsystems and simple fusion systems

Definition

Let $\mathcal{F}$ be a fusion system on the finite p-group S. A subsystem $\mathcal{E}$ on a strongly $\mathcal{F}$-closed subgroup T of S is

- *weakly normal* if $\alpha \operatorname{Hom}_{\mathcal{E}}(P, Q)\alpha^{-1} \subseteq \operatorname{Hom}_{\mathcal{E}}(\alpha(P), \alpha(Q))$ for any $P \leq Q \leq T$, and $\alpha \in \operatorname{Hom}_{\mathcal{F}}(Q, S)$.
- *strongly normal* or just *normal* if in addition every $\varphi \in \operatorname{Aut}_{\mathcal{E}}(T)$ extends to a morphism $\widetilde{\varphi} \in \operatorname{Aut}_{\mathcal{F}}(TC_S(T))$ such that $[C_S(T), \widetilde{\varphi}] \leq Z(T)$.

Definition

A fusion system is *simple* if it has no nontrivial strongly normal subsystems.

Theorem (Craven)

Let $\mathcal{F}$ be a fusion system and $\mathcal{E}$ a subsystem. If $\mathcal{E}$ is a weakly normal in $\mathcal{F}$, then $O^{p'}(\mathcal{E})$ is normal in $\mathcal{F}$.

Centralizers and components

Definition

Given a fusion system $\mathcal{F}$ on S and a *normal* subsystem $\mathcal{E}$ on the strongly $\mathcal{F}$-closed subgroup T of S, let $C_S(\mathcal{E})$ be the largest subgroup of $C_S(T)$ such that $C_{\mathcal{F}}(C_S(\mathcal{E}))$ contains $\mathcal{E}$ as a subsystem.

- $C_S(\mathcal{E})$ is strongly $\mathcal{F}$-closed; acts more like a "linking system centralizer".

Definition

The fusion system $\mathcal{F}$ is *quasisimple* if $O^2(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}/Z(\mathcal{F})$ is simple. A *component* of $\mathcal{F}$ is a quasisimple subnormal subsystem.

Definition

The *generalized Fitting subsystem* $F^*(\mathcal{F})$ of a fusion system $\mathcal{F}$ is the commuting product of the fusion systems of the p-group $O_p(\mathcal{F})$, and $E(\mathcal{F})$. Here $E(\mathcal{F})$ is a commuting product of the components of $\mathcal{F}$.

Theorem (Aschbacher)

$C_{\mathcal{F}}(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$.

Question

Why classify simple 2-fusion systems?

Answers:

- Far fewer simple 2-fusion systems, easier to identify.
- No cores! May simplify parts of the CFSG.
- Structured search for new exotic 2-fusion systems.

The Baumann Dichotomy

Definition

The *Thompson subgroup* $J(S)$ (for us) of a finite p-group S is the subgroup generated by the elementary abelian subgroups of S of maximal rank. The *Baumann subgroup* $\mathrm{Baum}(S)$ of S is defined as $C_S(\Omega_1 ZJ(S))$.

Definition

Let $\mathcal{F}$ be a fusion system on a 2-group S.

- $\mathcal{F}$ is of *Baumann characteristic 2-type* if for every fully $\mathcal{F}$-normalized subgroup $U \leq S$ with $\mathrm{Baum}(S) \leq N_S(U)$, $N_{\mathcal{F}}(U)$ is constrained. I.e. $E(N_{\mathcal{F}}(U)) = 1$.
- $\mathcal{F}$ is of *Baumann component type* if there exists a fully $\mathcal{F}$-centralized involution $x \in S$ such that $\mathrm{Baum}(S) \leq C_S(x)$ and $E(C_{\mathcal{F}}(x)) \neq 1$.

Theorem (Dichotomy Theorem)

A 2-fusion system is either of Baumann characteristic 2-type or of Baumann component type.

Fusion systems of Baumann component type

Roadmap for classification of simple 2-fusion systems of Baumann component type:

- Classical involution theorem (in progress: Aschbacher's quaternion fusion packets).
- Prove if an involution centralizer in $\mathcal{F}$ has a quaternion fusion packet, so does $\mathcal{F}$ or $\mathcal{F}$ is a Solomon system.
- Component theorem (standard components, Aschbacher's tightly embedded subsystems).
- Standard form problems for groups of Lie type in even characteristic, $L_2(q)$, alternating groups, several sporadic groups, and the Solomon systems.

Theorem (L.)

Let $\mathcal{F}$ be a fusion system on a 2-group S with $O^2(\mathcal{F}) = \mathcal{F}$ and $O_2(\mathcal{F}) = 1$. Suppose that $x \in S$ an involution such that

- *$C_{\mathcal{F}}(x)$ on $T = C_S(x)$ has a component $\mathcal{K}$ isomorphic to $\mathcal{F}_2(L_2(q^2))$ for some odd q,*
- *$T = C_S(x)$ contains the Baumann subgroup of S, and*
- *$Q = C_T(\mathcal{K})$ is cyclic.*

Then $\mathcal{F}$ is the 2-fusion system of $L_4(q)$.

Full disclosure: One last case is left to be proved: S is of 2-rank 4, $Q = \langle x \rangle$ is of order 2, and P (the Sylow subgroup of $\mathcal{K}$, a nonabelian dihedral group) is of order at least 16. It is expected that this will result in a contradiction, giving the above Theorem.

Key ingredients in the proof:

- Reduced and tame fusion systems (Andersen-Oliver-Ventura)
- Thompson transfer lemma for fusion systems + 2-group/fusion analysis

Pinning down $T\mathcal{K}/Q$

The hypotheses amount to supposing that

$$F^*(C_{\mathcal{F}}(x)) = Q \times \mathcal{K}.$$

Group case: TK/Q is a subgroup of $\mathrm{Aut}(K)$ containing K.

Example

Let $G = S_6$, $\mathcal{F} = \mathcal{F}_2(G)$ on T, $f \in G$ a transposition. Then f induces a trivial automorphism of $\mathcal{K} = \mathcal{F}_2(A_6)$, but $f \notin C_T(\mathcal{K}) = 1$.

We do know that $T\mathcal{K}/Q$ is a fusion system whose reduction (O^2) is isomorphic to $\mathcal{K}$, a fusion system of a finite group $K = L_2(q^2)$ which is *tame*. Therefore by Andersen-Oliver-Ventura,

Proposition

$T\mathcal{K}/Q$ is isomorphic to the fusion system of a subgroup of $\mathrm{Aut}(K)$ containing K.

This allows one to begin to get ahold on the structure of the centralizer $T = C_S(x)$.

Thompson's original lemma

Lemma (Thompson)

Suppose G is a finite group and S a Sylow 2-subgroup of G. Let $T < S$ be a maximal subgroup and u an involution in $S - T$. Then either u has a G-conjugate in T or else $u \notin O^2(G)$.

Proof.

The element u fixes an odd number of cosets of S in G, so the transfer map on u

$$\mathrm{tr}_S^G(u) = \prod_{ugS=gS} g^{-1}ug \neq 1 \bmod T.$$

□

Extensions:

- One can relax to S/T cyclic and take u of least order in $S - T$. (Harada?)
- One can choose the G conjugate of u to be "extremal" (or "fully centralized") (Goldschmidt)
- Can relax to S/T abelian if the set of G-conjugates of u in $S - T$ have linearly independent images in $\Omega_1(S/T)$. (Lyons)

Characteristic bisets

Consider G as an S-S-biset. Then ${}_S G_S = \coprod_{g \in [S \backslash G / S]} SgS$, and

$$SgS \cong (S \times S)/\Delta^{c_g{-1}}_{S \cap S^g} \text{ as an } S \times S\text{-set} .$$

This decomposition gives rise to Mackey decomposition of the transfer map:

$$\begin{aligned} \mathrm{tr}_S^G(u) &= \prod_{g \in [S \backslash G/S]} g\, \mathrm{tr}^S_{S \cap S^g}(u) g^{-1} \quad \text{for } u \in S \\ &= \prod_{g \in I} c_{g^{-1}}(\mathrm{tr}^S_{S_g}(u)) \end{aligned}$$

Theorem (Linckelmann-Webb, Broto-Levi-Oliver)

To any saturated fusion system $\mathcal{F}$, there exists an S-S-biset Ω in which each each transitive sub-biset is of the form $(S \times S)/\Delta^{\gamma_i}_{S_i}$ for some $\gamma_i \in \mathrm{Hom}_{\mathcal{F}}(S_i, S)$ and in which the $\{\gamma_i\}_{i \in I}$ play the role of a "set of S-S double coset representatives of the fusion system $\mathcal{F}$".

Thompson transfer for fusion systems

Theorem (L.)

Let $\mathcal{F}$ be a saturated fusion system on the p-group S, and let T be a subgroup of S with S/T abelian. Let $u \in S - T$, let $\mathcal{I}$ be the set of fully centralized $\mathcal{F}$-conjugates of u in $S - T$, and let $T\mathcal{I}$ be the set of cosets determined by $\mathcal{I}$. Suppose that

- *u is of least order, and*
- *$T\mathcal{I}$ is linearly independent in $\Omega_1(S/T)$.*

Then either

- *u has a fully centralized $\mathcal{F}$-conjugate in T, or*
- *$u \notin \mathfrak{foc}(\mathcal{F})$.*

Thank you for listening!