3. $2F$-modules, Nearly quadratic modules

Definition 3.1. *Let G be a group and V be a faithful $\mathbb{F}_p$-module. If there is some elementary abelian p-subgroup A of G, $1 \neq A$ and*

$$|V : C_V(A)| \leq |A|^2,$$

then we call V a $2F$-module and A an offender.

F-modules are $2F$-modules. Hence we will be interested in $2F$-modules, which are not F-modules

These modules again occur naturally in the amalgam method, when dealing with the distance 1 case. In the last lecture we considered some module V, which then induced quadratic modules in $O_p(H_2)$. But this module V itself has the structure of a $2F$-module.

Lemma 3.2. *Let $F^*(G) = O_p(G) \neq 1$, $A \leq G$ be an elementary abelian normal subgroup of a Sylow p-subgroup S of G and $A \not\leq O_p(G)$. Then one of the following holds:*

(i) There is some $g \in G$ such that for $X = \langle A, A^g \rangle$ the following hold

(1) $X/O_p(X) \cong SL_2(p^f)$, or $p = 2$ and $X/O_2(X) \cong Sz(p^f)$ or $X/O_2(X)$ is dihedral of order $2u$, u odd.

(2) $Y = (A \cap O_p(X))(A^g \cap O_p(X)) \leq S$ is normal in X, $Y \neq A \cap O_p(X)$.

(3) If $X/O_p(X)$ is not dihedral, then $Y/A \cap A^g$ is a direct sum of natural $X/O_p(X)$-modules

(ii) There is some $g \in G$ such that $B = A^g \leq S$, $[B, A] \neq 1$ and

$$|A : C_A(B)| = |B : C_B(A)|.$$

The second case is the F-module case. So assume we are in (i).

We realize that

$$[Y, Y] \leq [A \cap Y, A^g \cap Y] \leq A \cap A^g$$

by (2). Hence $[Y, Y, Y] = 1$. We have

$$[A, Y, Y, Y] = 1.$$

So we say that Y acts cubically.

Furthermore set $q = p^f$ if $X/O_p(X)$ is not dihedral and $q = 2$ else. Then we see with (3) that $|Y/A \cap Y| = q^x$, for some x and $|A : A \cap Y| \leq q$.

This shows that

$$|A : C_A(Y)| \leq q^x q \leq |Y : Y \cap A|^2 = |Y : C_Y(A)|^2.$$

Hence Y is an offender as a $2F$-module. So we have a $2F$-module with a cubic offender.

If Y acts quadratically we see that $[Y, A \cap Y] = 1$ and so $|A : C_A(Y)| \leq q \leq |Y : C_Y(A)|$ and so Y is an F-module offender.

Hence the lemma above says that we have either an F-module or a $2F$-module with cubic but not quadratic offender.

Further with (3) we can see that

$$[A, Y]C_A(Y) = Y \cap A = [a\mathbb{F}_p, Y]C_A(Y)$$

for all $a \in A \setminus Y \cap A$.

Such modules with cubic offender A are called nearly quadratic by U. Meierfrankenfeld.

We will give a short idea how to prove the lemma above.

In fact the proof is dependent on the classification of the finite simple groups.

We have that A acts quadratically on $O_p(G)$), as

$$[O_p(G), A, A] \leq [S, A, A] \leq [A, A] = 1.$$

Hence every result on quadratic groups is available.

If $|A : A \cap O_p(G)| = p$, we get some $g \in G$ such that either $\langle A, A^g \rangle O_p(G)/O_p(G) \cong SL_2(p)$ or $p = 2$ and this group is dihedral.

So we may assume that $|A : A \cap O_p(G)| \geq p^2$.

Furthermore let us assume that A acts faithfully on some quasisimple group L in $G/O_p(G)$.

As quadratic groups are small if L is not of Lie type in characteristic p, this is an easy case by case analysis.

So let L be a group of Lie type in characteristic p. Choose a parabolic $\hat{P}$ of L such that $S \cap \hat{P}$ is a Sylow p-subgroup of $\hat{P}$. Let P be the preimage. If $A \not\leq O_p(P)$, we have the result by induction on the Lie rank.

Hence we may assume that A is contained in $O_p(P)$ for all such parabolics. These intersections are well known.

For example if $L \cong SL_n(p^r)$ this intersection is a root group. Then A is contained in a root group and then also in some $SL_2(p^r)$, which is generated by two conjugates of the root group.

Unfortunately there are other cases. Now we can investigate $\langle A^P \rangle \leq O_p(P)$. If this group is non abelian we get a conjugate B of A with

$$1 \neq [B, A] \leq B \cap A$$

and we are in case (ii).

Hence we have that $\langle A^P \rangle$ is abelian for all parabolics P. This then shows $p = 2$ and we can embed A into some $PSp_4(2^r)$ and some special analysis is needed.

The result about the modules and so on follows easily.

So from the amalgam point of view we will be interested in nearly quadratic $2F$-modules.

As in the other cases before the $2F$-modules for the automorphism groups of quasisimple groups are known (Gurlanick, Lawther, Malle).

In fact the F-module theorem is just a corollary, but of course using the classification of the finite simple groups.

As in the case of an F-module one can show that $|V| \leq |G|^2$ for $2F$-modules V of G with $C_V(G) = 1$. This shows that these modules must be rare.

For example if $F^*(G)$ is an alternating group A_n. Then by the Stirling formula we have that

$$\log_p(|G|) \sim n \log_p(n).$$

It is well known that for large n only the heart of the permutation module and its product with the sign character have dimension less than $n^3/2$.

In particular for n large enough these are the only possible examples.

Similar for groups of Lie type in cross characteristic one can use the Landazuri-Seitz bound of minimal irreducible representations to come down with a short list.

The groups occurring besides the groups of Lie type in defining characteristic p and alternating groups are

(i) $p = 2$: $3U_4(3)$, M_{12}, M_{22}, $3M_{22}$, M_{23}, M_{24}, J_2.

(ii) $p = 3$: $2A_5$, $2A_9$, $2L_3(4)$, $Sp_6(2)$, $2Sp_6(2)$, $2\Omega_8^+(2)$, M_{11}, $2M_{12}$.

Remarkably all $2F$-modules for the quasisimple groups also possess an offender A, which acts cubically.

Now as before we may ask, whether offenders have to normalize components. The following result is due to Meierfrankenfeld and Stellmacher

Theorem 3.3. *Let G be a finite group, V a faithful $\mathbb{F}_p$-module and K be a component. Suppose there is a p-subgroup A with $|A/C_A(K)| > 2$ acting nearly quadratically on V.*

Then $|A/N_A(K)| \leq 2$ and either $A \leq N_G(K)$ or $p = 2$ and $K \cong SL_n(2)$ or $SL_2(2^n)$.

If A is also a $2F$-module offender, then in the case of $K \cong SL_n(2)$ for $a \in A$ with $K^a \neq K$, we have that $|[V, a]| = 2^n$ and A induces a full transvection group on $[V, a]$.

Furthermore Meierfrankenfeld and Stellmacher determined 13 exceptional cases, where A does not act faithfully on some component.

To have a nearly quadratic offender A, which is not quadratic makes life very easy. To give an indication we prove the following lemma, which obviously is wrong for F-modules.

Lemma 3.4. *Let V be a nearly quadratic module for G with offender A, which does not act quadratically. Suppose $V = V_1 \oplus V_2$ with $[V_i, A] \leq V_i$, $i = 1, 2$. Then A centralizes one of the V_i.*

Proof. As A acts quadratically on $[V, A]C_V(A)$, we may assume that there is some $v \in V_1$ such that $v \notin [V, A]C_V(A)$.

Now we get that

$$[V, A]C_V(A) = [v, A]C_V(A) \leq V_1 C_V(A).$$

In particular

$$[V, A, A] = [V_1, A] \leq V_1.$$

Hence

$$[V_2, A, A] \leq V_1 \cap V_2 = 1.$$

So

$$V_2 \leq [V, A]C_V(A)$$

and then

$$[V_2, A] \leq V_2 \cap [V, A, A] \leq V_1 \cap V_2 = 1.$$

□

The following result has been proven by Meierfrankenfeld, Stellmacher, Stroth

Theorem 3.5. *Let M and P be subgroups of a group G, $O_p(M) = F^*(M)$ and $O_p(P) = F^*(P)$.*

Assume that M and P share a common Sylow p-subgroup S and that S is contained in a unique maximal subgroup of P but not normal in P (usually called minimal parabolic).

Let $O_p(M) = C_S(Y_M)$ (Y_M is the largest elementary abelian normal p-subgroup of M such that $O_p(M/C_M(Y_M)) = 1$.)
and (M, P) be an amalgam (this basically means $O_p(\langle M, P\rangle) = 1$).

Set $V = \langle Y_M^P \rangle$. Then one of the following holds *(i)* $Y_M \not\leq O_p(P)$.

(ii) Y_M *is an* F*-module for* $M/C_M(Y_M)$.

(iii) The dual of Y_M *is an* F*-module for* $M/C_M(Y_M)$ *(If* Y_M *is irreducible it is an* F*-module)*

(iv) Y_M *is a* $2F$*-module with quadratic offender and* P *induces more than one nontrivial chief factor in* V.

(v) P *has exactly one nontrivial chief factor in* V, $O_p(M) \cap O_p(P)$ *is normal in* P *and*

$$[V, O^p(P)] \leq Z(O_p(P)).$$

This to a certain extend is the basis for proving the structure theorem in the revision of the classification of the finite simple groups.

Now using the classification (i.e. K_p-group assumption) there is the following result

Theorem 3.6. *Let G be a K_p-group (i.e. any simple composition factor of any p-local subgroup is in the list of simple groups) of local characteristic p ($O_p(P) = F^*(P)$ for any p-local P) with $O_p(G) = 1$.*

Let S be a Sylow p-subgroup of G. Then either there is exactly one maximal p-local containing S or there is some maximal p-local H with $S \leq H$ and Y_H is $2F$-module for H with cubic offender, or the dual of Y_H is an F-module.

In the second case any nontrivial chieffactor W of Y_H for $F^*(H/C_H(Y_H))$ is an F-module for $N_H(W)/C_H(W)$.

M. Aschbacher and S. Smith proved a special case of this in the quasithin paper.

There is a related result due to U. Meierfrankenfeld and B. Stellmacher, which does not use the classification.

Theorem 3.7. *Let G be a finite group of parabolic characteristic p and S be a Sylow p-subgroup. If S is contained in at least two maximal p-local subgroups of G, then there is a maximal p-local subgroup of G such that Y_M admits a nearly quadratic offender A.*

We now will give some ideas about relations with the revision of the classification.

From now on we are given a simple group and a prime p. There are two possible restrictions:

(i) G is of local characteristic p: If P is a nontrivial p-subgroup of G, then $O_p(N_G(P)) = F^*(N_G(P))$.

(ii) G is of parabolic characteristic p: If P is a nontrivial p-subgroup of G, which is normal in some Sylow p-subgroup, then $O_p(N_G(P)) = F^*(N_G(P))$.

Most result so far have been obtained for local characteristic p.

We further assume that G is a K_p-group, and there are at least two maximal p-local subgroups of G containing a given Sylow p-subgroup S.

Then the aim is to determine the structure of the maximal p-local subgroups of G containing S. Here the methods described so far play a major role.

Many results of this type have been obtained for groups of local characteristic p.

A group in Halle is trying to prove these results for parabolic characteristic p (at least if $p = 2$.)

If we know the structure of these p-locals, we then consider the subgroup

$$H = \langle P \mid S \leq P, P \text{ some } p - \text{local subgroups} \rangle.$$

In the generic case the group H will be a group of Lie type in characteristic p.

The problem then is to prove that G is a group of Lie type in characteristic p, which means $G = H$.

The other cases have been treated by several authors, including U. Meierfrankenfeld, Ch. Parker, M. Schmidt, S. Astill and myself.

So let us deal with the problem that we have a group G containing a subgroup H such that H contains a Sylow p-subgroup S and $N_G(U) \leq H$ for all $1 \neq U$, which are normal in S. Assume furthermore that $F^*(H)$ is a group of Lie type in characteristic p.

We try to prove that $G = H$.

For this we try to prove that H is strongly p-embedded, which means that $N_G(U) \leq H$ for all $1 \neq U \leq S$.

If we have achieved this then for $p = 2$ we have a result of Bender, which gives the conclusion.

If $p > 2$, and mild conditions on the Lie rank (let us say Lie rank at least three in this case), then a result of Parker and myself also shows $H = G$, assuming additionally that G is a K_2-group. The basic idea is to produce a classical involution and then use Aschbacher's theorem.

Hence the problem still is to prove that H is strongly p-embedded.

For groups of local characteristic p this has been done by M. Salarian and myself (here we did not assume K_p-group)

Here the basic idea of the proof is the following.

We choose a p-local P such that $P \not\leq H$ but $P \cap H$ contains a Sylow p-subgroup T of P. In this group we now choose a subgroup L which is minimal with respect to $T \leq L$ but $L \not\leq H$.

First of all as P is a p-local we have that

$$C_S(T) \leq P \text{ and so } C_S(T) \leq T.$$

This shows

$$Z(S) \leq T.$$

As G is of local characteristic p, we have that

$$C_L(O_p(L)) \leq O_p(L) \text{ and so } Z(S) \leq O_p(L).$$

Now choose $g \in L$ and $r \in Z(S)^\sharp$. Then

$$r^g \in O_p(L) \leq T \leq H.$$

So we investigate possible G-fusion of central elements in H.

For this we need the structure of L, which is given by a result of D. Bundy, N. Hebbinghaus and B. Stellmacher

In fact L cannot be factorized as $N_L(T)C_L(Z(T))$, so $L/O_p(L)$ possesses an F-module.

Assume that additionally H controls G-fusion of the elements in $Z(S)$, then we have some $h \in H$ with

$$r^g = r^h.$$

Then

$$r = r^{hg^{-1}}.$$

In particular $hg^{-1} \in C_G(r) \leq H$ and so also $g \in H$. As g was an arbitrary element from L, we get the contradiction $L \leq H$.

Hence so far there are two main ingredients :

(i) G is of local characteristic p

(ii) H controls fusion of elements in $Z(S)$.

M. Grimm tries to drop the first assumption for $p = 2$. But we have to pay for it by G is a K_2-group.

What about (ii)?

There is a well established tool when dealing with fusion, the *Alperin Fusion Theorem*.

This basically says that fusion is controlled by the $N_G(U)$, $1 \neq U \leq S$.

There are some restriction on U, like

$$C_{N_G(U)}(O_p(N_G(U))) \leq O_p(N_G(U))$$

and $N_S(U)$ is a Sylow p-subgroup of $N_G(U)$, but this does not help.

To investigate these groups is exactly the approach which was used.
In principal we just showed that H controls fusion and this in case of local characteristic p is enough. But it is not enough in general.

Now the connection with fusion systems comes into the game. Recently M. Aschbacher proved a nice result on saturated fusion systems on finite 2-groups.

Theorem 3.8. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S, such that $\mathcal{F}$ is a local CK-system of characteristic 2-type. Let $\mathcal{U}$ be the set of nontrivial normal subgroups U of S such that $C_S(U) \leq U$ and $O_2(\mathrm{Aut}_{\mathcal{F}}(U)) = \mathrm{Inn}(U)$. Then either*

(1) $\mathcal{F} = \langle N_{\mathcal{F}}(U) : U \in \mathcal{U} \rangle$ is generated by normalizers of members of $\mathcal{U}$, or

(2) $\mathcal{F}$ is an obstruction to pushing up at the prime 2.

This has the following corollary

Theorem 3.9. *Let G be a finite K_2-group of local characteristic 2 and S be a Sylow 2-subgroup. Then one of the following holds:*

(i) The subgroups $N_G(U)$, such that $C_S(U) \leq U$, U normal in S and $U = O_2(N_G(U))$ control fusion in S.

(ii) S is dihedral and $G \cong L_2(p)$, where p is a Fermat or Mersenne prime.

(iii) S is semidihedral of order 16 and $G \cong M_{10}$, M_{11}, or $L_3(3)$.

(iv) $|S| = 32$ and $G \cong \mathrm{Aut}(L_3(3))$.

(v) $|S| = 2^7$ and $G \cong J_3$.

This result serves our purpose at least for $p = 2$ (paying with K_2).

I think it is a very remarkable result and one should think about generalizations:

- Extend it to odd primes p

- Drop local characteristic 2. In the revision of the classification due to D. Gorenstein, R. Lyons and R. Solomon, local characteristic 2 has been removed by groups of even type.

  In a joint paper with K. Magaard we classified all groups of even type, which are not of parabolic characteristic 2 (paying with K_2).

So replace local characteristic 2 by parabolic characteristic 2, even characteristic in Aschbacher's language. (or do it for all primes with parabolic characteristic p).

This means probably to prove Aschbacher's result for saturated fusion systems of parabolic characteristic p, or at least of even characteristic.