2. Quadratic Modules

We have seen quadratic modules in the last lecture. We will repeat the definition

Definition 1.4. *(quadratic). Let G be a group and V be a faithful $\mathbb{F}_p$-module. If there is some $A \leq G$ with $[V, A] \neq 1 = [V, A, A]$, we say that V is a quadratic module and A is an offender.*

Sometimes one also asks $G = \langle A^G \rangle$.

Last time we asked for an elementary abelian p-subgroup A.

Lemma 2.1. *Let V be a faithful quadratic module for G over $\mathbb{F}_p$ with offender A, then A is an elementary abelian p-group.*

Proof. Choose $a, b \in A$ and $v \in V$. We have

$$v^a = vu_a, v^b = vu_b.$$

As

$$[u_a, a] = [u_b, a] = 1 = [b, u_b] = [b, u_a],$$

we see

$$v^{aba^{-1}b^{-1}} = vu_a u_b^{a^{-1}b^{-1}} = v.$$

Hence $[V, A'] = 1$ and so A is abelian.

If $a^p = b$, then

$$v^b = v^{a^p} = vu_a^p = v.$$

Hence also $[V, b] = 1$, so $b = 1$.

In particular A is elementary abelian. □

As involutions always act quadratically, we will assume $|A| > 2$ in all cases.

We have seen that quadratic modules show up in connection with F-modules. But there is also some occurrence in connection with the amalgam method.

Again we will consider two subgroups H_1, H_2 of a group G.

But now we will assume that they share a Sylow p-subgroup P.

Further $F^*(H_i) = O_p(H_i)$, $i = 1, 2$.

We will assume that there is a normal subgroup V_1 contained in $\Omega_1(Z(O_p(H_1)))$ such that

$$C_{H_2}(V_1) \leq C_{H_1}(V_1)(= O_p(H_1)).$$

Now we investigate the so called coset graph

$$\Gamma = \Gamma(H_1, H_2).$$

The vertices of this graph are the cosets of H_1 and H_2 and two such cosets are connected by an edge if they contain a coset of $H_1 \cap H_2$.

In particular the stabilizers of vertices are conjugates of H_1, H_2, respectively.

Now we choose $\alpha \in \Gamma$ such that $V_1 \leq H_\alpha$ but there is a neighbor β of α such that $V_1 \not\leq H_\beta$.

Furthermore we choose α such that the distance in Γ to 1 is minimal.

If $[V_1, V_\alpha] \neq 1$, we will end up with the situation of the last lecture.

So assume that $[V_1, V_\alpha] = 1$. For example this happens if $\Omega_1(Z(P))$ is normal in H_2.

We consider

$$W_\alpha = \langle V_\beta \mid \beta \in \Delta(\alpha) \rangle.$$

If the minimal distance was at least three, we get that W_α is elementary abelian.

Now we have some neighbor δ of 1 such that

$$d(1, \alpha) - 1 = d(\delta, \alpha).$$

In particular $W_\alpha \leq H_\delta$. As $V_1 \leq W_\delta$ we see that

$$[W_\alpha, V_1, V_1] \leq [W_\alpha, W_\delta, V_1] \leq [W_\delta, V_1] = 1.$$

So V_1 acts quadratically on W_α. Recall that by the general assumption on $C_{H_2}(V_1)$, we have that V_1 acts nontrivially on W_α.

Assume now $d(1, \alpha) = 1$. Then we may choose notation such that $\alpha = 2$ and so $V_1 \not\leq O_p(H_2)$.

In particular V_1 acts nontrivially on some nontrivial composition factor of H_2 in $O_p(H_2)$. As

$$[O_p(H_2), V_1, V_1] \leq [V_1, V_1] = 1,$$

we again come down with a quadratic module.

As in the last lecture we try to classify groups possessing a quadratic module.

Assume first that p is odd. There is a result due to Glauberman

Lemma 2.2. *Let p be odd and a be a p-element in G, which acts quadratically on a faithful $\mathbb{F}_p$-module V. If $a \notin O_p(G)$, then there is some conjugate a^g of a such that $\langle a, a^g \rangle / O_p(\langle a, a^g \rangle) \cong SL_2(p)$.*

This now shows

Lemma 2.3. *Let V be a quadratic module for G and a be a p-element acting quadratically. Assume $O_p(G) = 1$. If $p \geq 5$, then $[a, F(G)] = 1$ and $[a, K] \leq K$ for any component K of $E(G)$.*

Proof. Suppose that $a \notin F^*(G)$. Then by the lemma before we have a conjugate a^g, $g \in F^*(G)$, such that $\langle a, a^g \rangle \cong SL_2(p)$. But as $p \geq 5$, there is no normal subgroup of index p in $SL_2(p)$, a contradiction.

Hence $a \in F^*(G)$ and so $a \in E(G)$, as $O_p(G) = 1$.

Then $[F(G), a] = 1$ and a normalizes any component. □

In particular the investigation of quadratic modules and groups is reduced to those of the finite simple groups.

In this case there is some result which goes back to Thompson and which does not uses the classification of the finite simple groups.

Theorem 2.4. *Let G be a quasisimple group and V be a quadratic $\mathbb{F}_p$-module for $p \geq 5$, then G is a group of Lie type (different from type E_8) over a field of characteristic p*

The corresponding modules have been determined by A. Premet and I. Suprunenko, we will not list them.

But as a corollary we could prove the F-module theorem for $p > 3$ without using the classification of the finite simple groups.

A few words on the proof of this theorem above.

Let d be the minimal dimension of $[V, a]$ for a quadratically acting element $a \in G$. Then for such an a we define

$$E_a = \langle b \mid [V, b, b] = 1, [V, b] = [V, a], C_V(a) = C_V(b) \rangle.$$

These groups are called root groups.

Then Thompson showed that E_a is an elementary abelian p-group and any two conjugates of E_a, which do not generate a p-group generate

$$SL_2(q), q = |E_a|.$$

Hence basically he has the root subgroups of the groups of Lie type. Then using this he classified the groups of Lie type (nowadays one could also use Aschbachers classical involution theorem).

Assume now that $p = 3$.

Chat Ho then proved that the above theorem also holds, provided the root group has order at least 9.

So we deal with root groups of order three from now on.

Let $H = SL_2(9)$ and V be the natural module. Then 3-elements act quadratically.

Now we have a subgroup $SL_2(5)$. Here of course root groups have order 3. Furthermore G is generated by two root groups.

This is different to the case $p > 3$!

Ho proved

Lemma 2.5. *If* $\langle a\rangle$, $\langle b\rangle$ *are two root groups. Then* $H = \langle a, b\rangle$ *is isomorphic to an abelian group, an extraspecial group of order* 27, $SL_2(3)$, $SL_2(5)$ *or* $SL_2(3)\times\mathbb{Z}_3$.

Ho then classified those quasisimple groups, where the case $SL_2(3)\times\mathbb{Z}_3$ never occurs. He proved

Theorem 2.6. *Let* G *be a quasisimple group and* V *be a quadratic* $\mathbb{F}_3$*-module with root groups of order* 3*. Suppose there is some root group* A *such that* $\langle A, B\rangle \not\cong \mathbb{Z}_3\times SL_2(3)$ *for any root group* B*. Then either* G *is a group of Lie type in characteristic* 3 *or* $G\cong GU(n,2)$, $2\Omega_8^+(2)$, $2Sp_6(2)$, $2G_2(4)$, $2A_n$, $2J_2$, $2Suz$ *or* $2Co_1$.

Lateron Andy Chermak using the classification of the finite simple groups proved the same result but without the restriction on the root groups.

Theorem 2.7. *Let G be a known quasisimple group and V be a quadratic $\mathbb{F}_p$-module, p odd. Then either G is a group of Lie type or one of the exceptional groups in the theorem above.*

Hence to have a proof of the F-module theorem, which does not use the classification of the finite simple groups, we just have to prove that $\mathbb{Z}_3 \times SL_2(3)$ never occurs.

But there is another question for $p = 3$. Is it enough to deal with quasisimple groups?

Lemma 2.8. *Let a be a quadratic 3-element and K be a component. Then $[K, a] \leq K$.*

Proof. Assume $K^a = K_1K_2K_3$. Then there is a prime $p > 3$, with p divides the order of K_1.

In particular a acts on a p-group P nontrivially.

But then by the Glauberman-Lemma we get $g \in P$ such that $\langle a, a^g \rangle \cong SL_2(3)$, which contradicts the fact that $P\langle a \rangle$ is a $\{3, p\}$-group. □

Hence we may assume that $[a, E(G)] = 1$.

So the quadratic offender A acts faithfully on $F(G)$.

Then for an irreducible module V and $G = \langle A^G \rangle$ we have the following result of Chermak

Theorem 2.9. *Suppose that A is a quadratic offender of G on an irreducible module V, $F(G) = F^*(G)$ and $G = \langle A^G \rangle$. Then we have $|A| = 3$ and $F(G) = Z(G)X$, where X is extraspecial of width n and $|Z(G)| \leq 4$. Further $G/F(G) \cong$ $\mathrm{Alt}(2n+1)$, $\mathrm{Alt}(2n+2)$, $GU(n,2)$, $\Omega^{\pm}_{2n}(2)$ or $Sp_{2n}(2)$ and $F(G)/Z(F(G)$ is the natural module.*

Now we also can give a proof of Lemma (1.6) from the last lecture, at least for p odd.

Lemma 1.6. *Let $G = UA$, where A is an elementary abelian p-group and U is some p' group. Let V be some irreducible faithful $\mathbb{F}_p$-module for G on which A induces an F-module offender. Then $p = 2$ or 3 and $|V : C_V(A)| = |A|$.*

As we have seen we have $p = 3$. Furthermore $[F(G), A]$ is a 2-group T. Let C be a critical subgroup in T, then we have that C is not abelian, in particular $[A, Z(C)] = 1$.

Now $C/Z(C)$ is a direct sum of $C_{C/Z(C)}(B)$, where B are the hyperplanes in A.

This now shows that we have a product

$$X_1 \cdots X_n \leq CA, X_i \cong SL_2(3)$$

and

$$[X_i, X_j] = 1 \text{ for } i \neq j$$

and A is a Sylow 3-subgroup.

We now may assume that $V = [V, X_1 \cdots X_n]$, so

$$V = \oplus_{i=1}^{n} [V, X_i].$$

But then $|[V, X_i] : C_{[V,X_i]}(A)| \leq 3$, as A is an F-module offender. In particular

$$|V : C_V(A)| = |A|$$

and A is generated by transvections on V.

We now come to the case of $p = 2$. Recall that we now have $|A| \geq 4$.

First of all we prove (1.6) for $p = 2$.

In this case we use Thompson's dihedral Lemma

Lemma 2.10. *Let A be an elementary abelian 2-group which acts faithfully on some p-group P, p odd, then PA contains some subgroup*

$$D = D_1 \times \cdots \times D_n,$$

which possesses A as a Sylow 2-subgroup, where the D_i are dihedral of order $2p$.

This implies $[V, D] = \oplus [V, D_i]$ by quadratic action. Then as above we get that

$$|[V, D_i] : C_{[V, D_i]}(A)| \leq 2.$$

Hence we have equality.

We now assume that A acts faithfully on $E(G)$

First of all we have the following Lemma, from which Lemma (1.5) follows.

Lemma 2.11. *Let K be a component of G and V be a $\mathbb{F}_2$–module for G with $[V, K] \neq 1$. Let A be a quadratic group on V, then one of the following holds*

(i) $[K, A] \leq K$

(ii) $A \neq N_A(K)$, $|A/C_A(K)| = 2$

(iii) $K \cong SL_2(2^k)$ *and* $|A/N_A(K)| = 2$

In (ii) and (iii) A is not a quadratic offender as an F-module on $[V, K]$ and $[V, K]$ involves at least two nontrivial K-modules.

So we now may assume that A normalizes a component K, i.e. we may assume that $F^*(G)$ is quasisimple.

Of course the groups of Lie type in characteristic 2 admit such modules. But to my knowledge there is no classification of the corresponding modules. There is one if A intersects a root group nontrivially but is not contained in this root group.

All now is joint work with U. Meierfrankenfeld.

When dealing with groups of Lie type in odd characteristic some result due to Alperin is helpful

Lemma 2.12. *Let G be a group A a quadratic fours group, which normalizes some subgroup E of odd order. Then $\bigcap_{a \in A^\sharp} [E, a] = 1$.*

This then implies that A cannot normalize a long root group in G. Hence it has to act nontrivially on the fundamental subgroups of a Sylow 2-subgroup containing A. Then it follows that A contains some involution z from this fundamental subgroup.

Lemma 2.13. *For $a \in A^\sharp$ set $R_a = \langle A^{C_G(a)} \rangle$. Then we have that*

(i) $[V, a, R_a] = [V, R_a, a] = 1$, *and*

(ii) If $b \in R_a$, $b \neq a$ an involution. Then $\langle a, b \rangle$ is a quadratic fours group.

Proof. (i) We have $[V, a, A] = 1$. Hence also

$$1 = [V^{C_G(a)}, a^{C_G(a)}, A^{C_G(a)}] = [V, a, R_a].$$

As $[a, R_a] = 1$, the other relation follows with the 3-subgroup lemma.

By (i) we have that

$$[V, a, b] = 1 = [V, b, a].$$

As $[V, a, a] = 1 = [V, b, b]$, we get that

$$[V, a, \langle a, b \rangle] = 1 = [V, b, \langle a, b \rangle]$$

and so $\langle a, b \rangle$ is quadratic. □

Now we have that R_z contains a central product of n subgroups $SL_2(q)$, q odd.

If $n > 2$, there is some involution b in this product, which centralizes one of these groups. Hence by the Lemma $\langle z, b \rangle$ also is quadratic.

But as said before, a quadratic fours group cannot centralize a fundamental group. Hence $n \leq 2$ and then it is fairly easy to analyze the groups and to get

Theorem 2.14. *Assume that $F^*(G)/Z(F^*(G))$ is a group of Lie type in odd characteristic, which is not also a group of Lie type in characteristic 2, then $F^*(G) \cong 3U_4(3)$ or $3^2U_4(3)$. If V is an irreducible module for $F^*(G)$, then $\dim V = 12$ and $F^*(G) \cong 3U_4(3)$.*

So we are left with the sporadic groups and the alternating groups. Here are the corresponding results

Theorem 2.15. *Let $F^*(G)$ be a central extension of a sporadic group, $G = \langle A^G \rangle$. Then $G \cong M_{12}$, $\mathrm{Aut}(M_{12})$, $\mathrm{Aut}(M_{22})$, $3M_{22}$, M_{24}, J_2, Co_1, Co_2 or $3Suz$. Also the list of irreducible V is given.*

Lemma 2.16. *Let G be a group, V be a faithful $\mathbb{F}_2$-module and E a fours group, which is not contained in $O_2(G)$. Suppose that for $e \in E^\sharp$ the following holds*

(i) $C_G(e)$ *is maximal, and*

(ii) $E \not\leq Z(C_G(e))$.

Then there are $e, f \in E^\sharp$, $e \neq f$, with

$$[V, e] \cap [V, f] \neq 1.$$

Proof. Choose $e \in E^\sharp$ such that $|C_V(e)|$ is maximal. Now pick $f \in E^\sharp$, $f \neq e$.

If $[C_V(e), f] \neq 1$, then $[V, f] \cap [V, ef] \neq 1$.

So we may assume that $[C_V(e), f] = 1$. By maximality of $C_V(e)$ we have $C_V(e) = C_V(f)$.

Now this group is invariant under $\langle C_G(e), C_G(f) \rangle$.

By (i) and (ii) we conclude that $G = \langle C_G(e), C_G(f) \rangle$, so $C_V(e)$ is G-invariant.

As $[V, e] \leq C_V(e) \geq [V, f]$, we have

$$[V, E] \leq C_V(E) \text{ and so } [V, \langle E^G \rangle] \leq C_V(E)$$

$\langle E^G \rangle$ acts quadratically on V. Then $\langle E^G \rangle$ is a 2-group, which contradicts $E \not\leq O_2(G)$. □

Lemma 2.17. *Let G be a group, V be a faithful $\mathbb{F}_2$-module and E be a quadratic fours group. If $\langle R_e, R_f \rangle = G$, for some $e, f \in E^\sharp$, then*

$$[V, e] \cap [V, f] \leq C_V(G).$$

Proof. We have that $[V, e, R_e] = 1 = [V, f, R_f]$. Hence $[V, e] \cap [V, f]$ is centralized by $\langle R_e, R_f \rangle = G$. □

Lemma 2.18. *There in no module for* $G = M_{11}$*, which admits a quadratic fours group.*

Proof. We may assume that V is irreducible. Let A be a fours group. If $a \in A^\sharp$, then

$$C_G(a) \cong GL_2(3)$$

is a maximal subgroup of G. Further

$$R_a = C_G(a).$$

By the first lemma we have $b \in A^\sharp$, $a \neq b$, with

$$[V, a] \cap [V, b] \neq 1.$$

By the second lemma we have that

$$[V, a] \cap [V, b] \leq C_V(G),$$

a contradiction. □

Theorem 2.19. *Let* $F^*(G)/Z(F^*(G)) \cong \mathrm{Alt}(n)$*.*
If V *is irreducible, then either* V *is the natural module or the spin module or*
$F^*(G) \cong 3\mathrm{Alt}(6)$ *or* $3\mathrm{Alt}(7)$ *and*
V *is a* 6*-dimensional or* 12*-dimensional module.*