Exercises on p-local finite groups

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Let $\mathcal{F}$ be a fusion system on the p-group S. Recall that an *abstract transporter system on* $\mathcal{F}$ is a category $\mathcal{T}$ whose objects are a collection of subgroups of S, closed under $\mathcal{F}$-conjugacy and overgroups, together with a pair of functors

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{T})}(S) \xrightarrow{\delta} \mathcal{T} \xrightarrow{\pi} \mathcal{F}$$

that satisfy

(A) On objects, δ is the identity and π is the inclusion. Moreover, for each $P, Q \in \mathrm{Ob}(\mathcal{T})$, the group

$$E(P) := \ker[\pi_{P,P} : \mathcal{T}(P) \to \mathcal{F}(P)]$$

acts freely (i.e., with trivial stabilizers) on $\mathcal{T}(P,Q)$ by right composition, and $\pi_{P,Q} : \mathcal{T}(P,Q) \to \mathcal{F}(P,Q)$ is the orbit map of this action. In particular, π is surjective on morphisms. Also, $E(Q)$ acts freely on $\mathcal{T}(P,Q)$ by left composition.

(B) The functor δ is injective on morphisms, and for any $g \in N_S(P,Q)$, the composite $\pi_{P,Q} \circ \delta_{P,Q}$ sends g to $c_g \in \mathcal{F}(P,Q)$.

(C) For all $\mathfrak{g} \in \mathcal{T}(P,Q)$ and $a \in P$, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{g}} & Q \\ {\scriptstyle \delta_{P,P}(a)}\downarrow & & \downarrow{\scriptstyle \delta_{Q,Q}(\pi(\mathfrak{g})(a))} \\ P & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

commutes in $\mathcal{T}$.

(I) $\delta_{S,S}(S) \in \mathrm{Syl}_p(\mathcal{T}(S))$.

(II) For any $\mathfrak{g} \in \mathcal{T}(P,Q)_{\mathrm{Iso}}$ and normal overgroups $P \trianglelefteq P' \leq S$ and $Q \trianglelefteq Q' \leq S$ such that

$$\mathfrak{g} \circ \delta_{P,P}(P') \circ \mathfrak{g}^{-1} \leq \delta_{Q,Q}(Q')$$

there is an element ("extension of $\mathfrak{g}$") $\mathfrak{g}' \in \mathcal{T}(P',Q')$ such that

$$\begin{array}{ccc} P' & \xrightarrow{\mathfrak{g}'} & Q' \\ {\scriptstyle \delta_{P,Q}(1)}\uparrow & & \uparrow{\scriptstyle \delta_{Q,Q'}(1)} \\ P & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

commutes in $\mathcal{T}$.

Exercise 1. Let $\mathcal{T}$ be an abstract transporter system (e.g., a centric linking system), and let $\mathfrak{i}$ denote any "inclusion" morphism, namely, any morphism of the form $\delta_{P,Q}(1)$ for $P \leq Q$. Let $\mathrm{Iso}(\mathcal{T})$ denote the set of isomorphisms of $\mathcal{T}$, and let $\sim$ denote the equivalence relation on $\mathrm{Iso}(\mathcal{T})$ generated by restriction. Explicitly, if $P \leq Q$ and we have isomorphisms $\mathfrak{g}_P \in \mathcal{T}(P, P')_{\mathrm{Iso}}$ and $\mathfrak{g}_Q \in \mathcal{T}(Q, Q')_{\mathrm{Iso}}$, we have $\mathfrak{g}_P \sim \mathfrak{g}_Q$ if the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\mathfrak{g}_Q} & Q' \\ {\scriptstyle \mathfrak{i}}\uparrow & & \uparrow{\scriptstyle \mathfrak{i}} \\ P & \xrightarrow[\mathfrak{g}_P]{} & P' \end{array}$$

commutes in $\mathcal{T}$. In general, $\sim$ is the symmetric, transitive closure of this relation.

(a) Suppose that $P \leq Q, R$ are objects of $\mathcal{T}$, and set $A = Q \cap R$. Given $\mathfrak{g}_Q, \mathfrak{g}_R \in \mathrm{Iso}(\mathcal{T})$ two isomorphisms with sources Q and R, respectively, such that $\mathfrak{g}_Q\big|_P = \mathfrak{g}_R\big|_P =: \mathfrak{g}_P$, show that $\mathfrak{g}_Q\big|_A = \mathfrak{g}_R\big|_A$, and that this morphism is an extension of $\mathfrak{g}_P$.

(b) Suppose that $P \trianglelefteq Q, R$ are objects of $\mathcal{T}$. Given $\mathfrak{g}_Q, \mathfrak{g}_R \in \mathrm{Iso}(\mathcal{T})$ two isomorphisms with sources Q and R, respectively, such that $\mathfrak{g}_Q\big|_P = \mathfrak{g}_R\big|_P =: \mathfrak{g}_P$, show that there is some $\mathfrak{g}_U \in \mathrm{Iso}(\mathcal{T})$ with source $U := \langle Q, R \rangle$ whose restrictions to P, Q, and R are the respective isomorphisms.

Hint: The extension axioms for transporter systems and fusion systems should both be helpful.

(c) Let $\mathfrak{g}_Q \in \mathrm{Iso}(\mathcal{T})$ be an isomorphism in $\mathcal{T}$ with source Q, and suppose that $\mathfrak{g}_Q$ has no proper extensions in $\mathcal{T}$. Show that if $\mathfrak{g}_R \in \mathrm{Iso}(\mathcal{T})$ is another isomorphism (with source R) such that $\mathfrak{g}_Q \sim \mathfrak{g}_R$, then in fact $R \leq Q$ and $\mathfrak{g}_R = \mathfrak{g}_Q\big|_R$.

Hint: Use that $\mathfrak{g}_Q$ does not have any proper extensions to conclude that we must only consider the case where we have $P \leq Q, R$ is an object of $\mathcal{T}$ and $\mathfrak{g}_Q\big|_P = \mathfrak{g}_R\big|_P$. Induct downward on the order of such a P.

Exercise 2. Let $\mathcal{T}$ be a transporter system.

(a) Define an interesting map θ from the set $\mathrm{Mor}(\mathcal{T})$ of morphisms of $\mathcal{T}$ to $\pi_1(|\mathcal{T}|, S)$ (here we view the object S of $\mathcal{T}$ as a basepoint for the fundamental group) that sends inclusions to the identity and compositions to multiplication.

(b) Show that the map θ from (a) is universal in the following sense: If $F : \mathcal{T} \to \mathcal{B}G$ is a functor from $\mathcal{T}$ to the classifying category of a discrete group that sends inclusions to the identity, there is a unique homomorphism $\varphi : \pi_1(|\mathcal{T}|, S) \to G$ such that

$$\begin{array}{ccc} \mathrm{Mor}(\mathcal{T}) & \xrightarrow{F} & G \\ {\scriptstyle \theta}\downarrow & \nearrow{\scriptstyle \varphi} & \\ \pi_1(|\mathcal{T}|, S) & & \end{array}$$

commutes. You may assume the result from topology that states that the image of θ generates $\pi_1(|\mathcal{T}|, S)$.

(c) Conclude with a description of $\pi_1(|\mathcal{T}|, S)$ in terms of generators and relations.

Exercise 3. Use the results of Exercise 1 to construct a partial group whose elements are the maximal isomorphisms of $\mathcal{T}$. Describe $\pi_1(|\mathcal{T}|, S)$ in terms of this partial group.