Main results

we assume that X is linearly ordered

Recall from yesterday:

Theorem

The following are equivalent for an $\mathcal{NT}$-module M:

- *M is a direct sum of free modules.*
- *M is projective.*
- *M is free as an Abelian group and exact.*

Theorem

The following are equivalent for an $\mathcal{NT}$-module M:

- *M has a projective resolution of length 1.*
- *M has a projective resolution of finite length.*
- *M is exact.*
- *M is in the range of filtrated K-theory.*

Hence there are $\mathcal{NT}$-modules without a projective resolution of finite length, but these cannot arise as filtrated K-groups.

The totally ordered case

The setup

Let $X = \{1, \dots, n\}$, write $[a, b] := \{a, a+1, \dots, b\}$ and let

$$\mathbb{O}(X) := \{[1, k], k = 0, \dots, n\}.$$

- $\mathbb{LC}(X)^* = \{[a,b] \mid 1 \le a \le b \le n\}$
- For $n = 3$: $\mathbb{LC}(X)^* = \{1, 12, 123, 2, 23, 3\}$
- $\mathcal{R}_{[1,k]} = i_k\mathbb{C}$
- We have $[a,b] = [1,b] \setminus [1,a-1]$, and the inclusion $[1,a-1] \subseteq [1,b]$ corresponds to an inclusion morphism $f_{ba}\colon i_b\mathbb{C} \to i_{a-1}\mathbb{C}$.
- The K-theory long exact sequence for

$$A([1,a-1]) \rightarrowtail A([1,b]) \twoheadrightarrow A([a,b])$$

must correspond to an exact triangle

$$\mathcal{R}_{[a,b]} \to \mathcal{R}_{[1,b]} \to \mathcal{R}_{[1,a-1]} \to \mathcal{R}_{[a,b]}[1].$$

The representing objects

Lemma

The mapping cone of f_{ba} is a model for $\mathcal{R}_{[a,b]}$. Thus

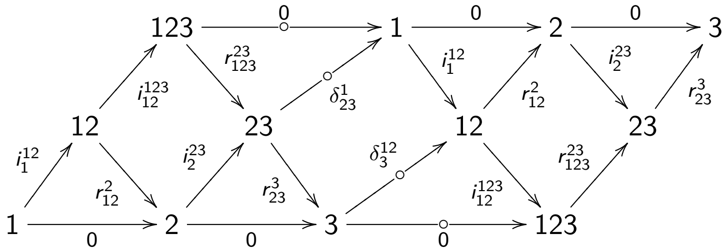
$$\mathcal{R}_{[a,b]}(Y) = \begin{cases} \mathcal{C}_0((0,1]) & a-1, b \in Y, \\ \mathcal{C}_0((0,1)) & a-1 \in Y,\ b \notin Y, \\ \mathbb{C} & a-1 \notin Y,\ b \in Y, \\ 0 & a-1, b \notin Y. \end{cases}$$

and $\mathcal{NT}(Z,Y) := \mathrm{KK}^X_*(\mathcal{R}_Y, \mathcal{R}_Z) \cong \mathrm{K}_*(\mathcal{R}_Z(Y))$ *is*

$$\mathcal{NT}([a,b],Y) = \begin{cases} 0 & a-1, b \in Y, \\ \mathbb{Z}^{\mathrm{odd}} & a-1 \in Y,\ b \notin Y, \\ \mathbb{Z}^{\mathrm{even}} & a-1 \notin Y,\ b \in Y, \\ 0 & a-1, b \notin Y. \end{cases}$$

Modules over $\mathcal{NT}$ for $n = 3$

Modules over $\mathcal{NT}$ consist of 6 Abelian groups together with maps as indicated in the following skew-periodic diagram:



- The diagram commutes.
- Arrows marked with a circle are of degree 1.
- Arrows marked zero denote zero maps.

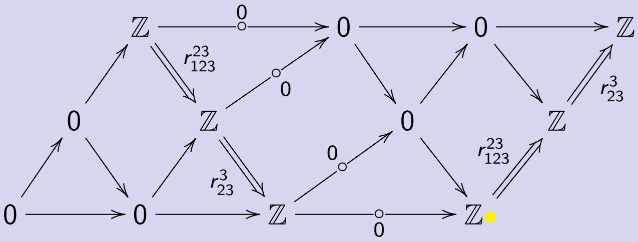
Definition (Free modules)

$$F(\mathcal{R}_Z) = \bigoplus_{Y \in \mathbb{LC}(X)^*} \mathrm{K}_*(\mathcal{R}_Z(Y)) = \bigoplus_{Y \in \mathbb{LC}(X)^*} \mathcal{NT}(Z, Y)$$

Free modules are projective

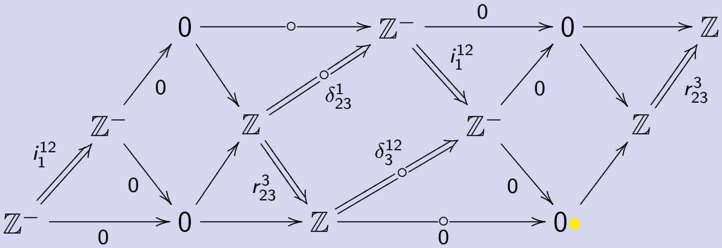
Example

For $n = 3$, the free module $F(\mathcal{R}_{123})$ looks as follows:



Example

For $n = 3$, the free module $F(\mathcal{R}_{23})$ looks as follows:



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The following are equivalent for an $\mathcal{NT}$-module M:

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The following are equivalent for an $\mathcal{NT}$-module M:

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- *M is in the range of filtrated K-theory.*

Proof.

(1)$\Longrightarrow$(2) is trivial and
(1)$\Longrightarrow$(4) follows from the fact that free modules are representable.
(2)$\Longrightarrow$(3) follows by iterating the two-out-of-three property of exactness for extensions.
(4)$\Longrightarrow$(3)is just excision for KK.
(3)$\Longrightarrow$(1)
Let $K \rightarrowtail P \twoheadrightarrow M$ be a module extension with projective P. Then K is exact because P and M are, and a free Abelian group because $K \subseteq P$ and P is free. These two conditions are sufficient for projectivity by previous theorem. □

Thus everything reduces to proving that exact, torsionfree $\mathcal{NT}$-modules are projective.

Characterisation of free modules

Question

Does the ring $\mathcal{NT}$ have some special features?

- $\mathcal{NT}$ is finitely generated as an Abelian group.
- $\mathcal{NT}$ is a split extension of a semisimple ring by a nilpotent ideal:

Definition and Lemma

Let $\mathcal{NT}_{\text{ss}} \subseteq \mathcal{NT}$ be $\bigoplus \mathcal{NT}(Y, Y)$.
Let $\mathcal{NT}_{\text{nil}} \subseteq \mathcal{NT}$ be $\bigoplus_{Y \neq Z} \mathcal{NT}(Y, Z)$.
Then

- $\mathcal{NT}_{\text{nil}}$ is a nilpotent ideal,
- $\mathcal{NT}_{\text{ss}}$ is a semi-simple subring, and
- $\mathcal{NT} = \mathcal{NT}_{\text{nil}} \rtimes \mathcal{NT}_{\text{ss}}$.

Free versus projective modules

Corollary

The projection $\mathcal{NT} \twoheadrightarrow \mathcal{NT}_{\mathrm{ss}}$ induces a bijection between isomorphism classes ofprojective modules over $\mathcal{NT}$ and $\mathcal{NT}_{\mathrm{ss}}$. The latter are just families of free $\mathbb{Z}/2$-graded Abelian groups $(G_Y)_{Y \in \mathbb{LC}(X)^*}$.
Any projective $\mathcal{NT}$-module is of the form $\mathcal{NT} \otimes_{\mathcal{NT}_{\mathrm{ss}}} P$ for a free $\mathcal{NT}_{\mathrm{ss}}$-module and hence a direct sum of free modules.

Lemma

Let M be an $\mathcal{NT}$-module that is free as an Abelian group. Then $M_{\mathrm{ss}} := \mathcal{NT}_{\mathrm{ss}} \otimes_{\mathcal{NT}} M$ is a free Abelian group as well, and $P := \mathcal{NT} \otimes_{\mathcal{NT}_{\mathrm{ss}}} M_{\mathrm{ss}}$ is a free $\mathcal{NT}$-module. There is a canonical map $P \to M$, which induces an isomorphism $P_{\mathrm{ss}} \to M_{\mathrm{ss}}$.

The Trivial Nakayama Lemma

A a unital ring

I a nilpotent ideal in A: $I^k = 0$ for some $k \in \mathbb{N}$

M an A-module

Lemma

If $I \cdot M = M$ or, equivalently, $(A/I) \otimes_A M \cong 0$, then $M = 0$.

Proof.

$I \cdot M = M \Longrightarrow 0 = 0 \cdot M = I^k \cdot M = M$

$(A/I) \otimes_A M \cong M/I \cdot M.$ □

Another criterion for projective modules

Lemma

Let $f\colon P \to M$ be a map between two $\mathcal{NT}$-modules.
Assume that $f_{\mathrm{ss}}\colon P_{\mathrm{ss}} \to M_{\mathrm{ss}}$ is an isomorphism and that P is projective.
Then f is surjective, and it is injective if and only if $\mathrm{Tor}_1^{\mathcal{NT}}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$.

Proof.

The Trivial Nakayama Lemma yields $\operatorname{coker}(f) = 0$ because

$$\operatorname{coker}(f)_{\mathrm{ss}} = \operatorname{coker}(f_{\mathrm{ss}}) = 0.$$

Let $K := \ker(f)$. Then there is an exact sequence

$$0 \to \operatorname{Tor}_1^{\mathcal{NT}}(\mathcal{NT}_{\mathrm{ss}}, M) \to K_{\mathrm{ss}} \to P_{\mathrm{ss}} \xrightarrow[\cong]{f_{\mathrm{ss}}} M_{\mathrm{ss}} \to 0.$$

$$K = 0 \iff K_{\mathrm{ss}} = 0 \iff \operatorname{Tor}_1^{\mathcal{NT}}(\mathcal{NT}_{\mathrm{ss}}, M) = 0.$$

□

The crucial step

- Let M be an exact $\mathcal{NT}$-module that is free as an Abelian group. We want to show that M is projective.
- We have constructed a projective $\mathcal{NT}$-module P and a map $f\colon P \to M$ such that f_{ss} is invertible.
- By the Trivial Nakayama Lemma, f is surjective.
- We get a module extension $K \rightarrowtail P \twoheadrightarrow M$.
- It remains to prove $K_{ss} = 0$.

Lemma

Let $h\colon K \to P$ be an injective $\mathcal{NT}$-module homomorphism. If K is exact, then $h_{ss}\colon K_{ss} \to P_{ss}$ is injective as well.

- This lemma applies because K is exact, and yields $K_{ss} = 0$ as needed, showing that M is projective.

Proof of the crucial lemma

Lemma

Let $h\colon K \to P$ be an injective $\mathcal{NT}$-module homomorphism. If K is exact, then $h_{\mathrm{ss}}\colon K_{\mathrm{ss}} \to P_{\mathrm{ss}}$ is injective as well.

Proof.

Let $x \in K(Y)$. We must show:

$$h(x) \in \mathcal{NT}_{\mathrm{nil}} \cdot P(Y) \quad \Longrightarrow \quad x \in \mathcal{NT}_{\mathrm{nil}} \cdot K(Y).$$

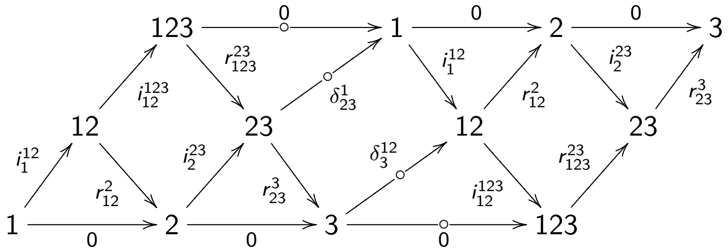
- There are one or two generators α_Y, β_Y of $\mathcal{NT}(?, Y)$ such that $(\mathcal{NT}_{\mathrm{nil}} \cdot P)(Y) = \operatorname{range} \alpha_Y + \operatorname{range} \beta_Y$.
- Let $\gamma_Y \in \mathcal{NT}(Y, ?)$ be the longest arrow out of Y. Then $\operatorname{range} \alpha_Y + \operatorname{range} \beta_Y \subseteq \ker \gamma_Y$,
  with equality for any exact module.

$\Longrightarrow$ $\gamma_Y(h(x)) = 0$

$\Longrightarrow$ $\gamma_Y(x) = 0$ because h is injective

$\Longrightarrow$ $x \in \mathcal{NT}_{\mathrm{nil}} \cdot K(Y)$ because K is exact. □

An example of the proof method



- Any arrow to $Y = 23$ factors through $\alpha_Y = i^{23}_2$ or $\beta_Y = r^{23}_{123}$, and the longest arrow out of Y is $\gamma_Y = \delta^{12}_3 r^3_{23}$.
- $\gamma_Y \circ \alpha_Y = 0$ and $\gamma_Y \circ \beta_Y = 0$
- For an exact module, $\operatorname{range} \alpha_Y = \ker r^3_{23}$ and

$$r^3_{23}(\operatorname{range} \beta_Y) = \operatorname{range} r^3_{123} = \ker \delta^{12}_3.$$

$\Longrightarrow$ $\ker \gamma_Y = \operatorname{range} \alpha_Y + \operatorname{range} \beta_Y$ for an exact module.

Further reading (see arxiv)

Meyer and Nest.
The Baum–Connes conjecture via localisation of categories.

Meyer and Nest.
Homological algebra in bivariant K-theory and other triangulated categories. I.

Meyer.
Homological algebra in bivariant K-theory and other triangulated categories. II.

Meyer and Nest.
C^*-Algebras over topological spaces: the bootstrap class.

Meyer and Nest.
C^*-Algebras over topological spaces: filtrated K-theory.