Let $\mathfrak{C}$ be an abelian category
We call a covariant functor $F\colon \mathfrak{T} \to \mathfrak{C}$ *homological* if

$$F(C) \to F(A) \to F(B)$$

is exact for any distinguished triangle

$$\Sigma B \to C \to A \to B.$$

We define $F_n(A) := F(\Sigma^n A)$ for $n \in \mathbb{Z}$.
Similarly, we call a contravariant functor $F\colon \mathfrak{T} \to \mathfrak{C}$ *cohomological* if $F(B) \to F(A) \to F(C)$ is exact for any distinguished triangle, and we define $F^n(A) := F(\Sigma^n A)$.

We will always be in the following situation.

1. $\mathfrak{C}$ is some abelian category equipped with a shift $\Sigma : \mathfrak{C} \to \mathfrak{C}$ ($\mathfrak{C}$ is stable).
2. Our functors $F : \mathfrak{T} \to \mathfrak{C}$ are homological and stable, i. e. commute with Σ.

We will do homological algebra relative to some ideal in $\mathfrak{T}$ wich satisfies the following property.

Definition

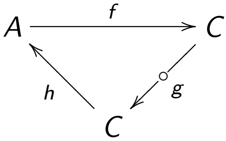
An ideal $\mathfrak{J} \subseteq \mathfrak{T}$ is called *stable* if the suspension isomorphisms $\Sigma\colon \mathfrak{T}(A, B) \xrightarrow{\cong} \mathfrak{T}(\Sigma A, \Sigma B)$ for $A, B \in\in \mathfrak{T}$ restrict to isomorphisms

$$\Sigma\colon \mathfrak{J}(A, B) \xrightarrow{\cong} \mathfrak{J}(\Sigma A, \Sigma B).$$

Definition

An ideal $\mathfrak{J} \subseteq \mathfrak{T}$ in a triangulated category is called *homological* if it is the kernel of a stable homological functor.

Before continuing with definitions, recall the picture of distinguished triangle in $\mathfrak{T}$.



Definition

Let $\mathfrak{T}$ be a triangulated category and let $\mathfrak{J} \subseteq \mathfrak{T}$ be a homological ideal. Let $f\colon A \to B$ be a morphism in $\mathfrak{T}$ and embed it in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

- We call f $\mathfrak{J}$ *monic* if $h \in \mathfrak{J}$.
- We call f $\mathfrak{J}$ *epic* if $g \in \mathfrak{J}$.
- We call f an $\mathfrak{J}$ *equivalence* if f is both $\mathfrak{J}$ monic and $\mathfrak{J}$ epic or, equivalently, if $g, h \in \mathfrak{J}$.
- We call f an $\mathfrak{J}$ *phantom map* if $f \in \mathfrak{J}$.

An object $A \in\in \mathfrak{T}$ is called $\mathfrak{J}$ *contractible* if $\mathrm{id}_A \in \mathfrak{J}(A, A)$.

An exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathfrak{T}$ is called $\mathfrak{J}$ *exact* if $h \in \mathfrak{J}$.

Consider a chain complex $C_\bullet = (C_n, d_n)$. For each $n \in \mathbb{N}$, we may embed the map d_n in an exact triangle

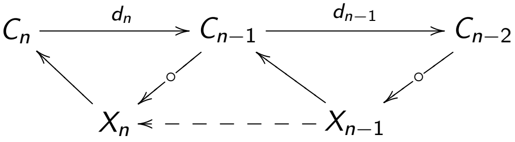
$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{f_n} X_n \xrightarrow{g_n} \Sigma C_n, \tag{1.1}$$

which is determined uniquely up to (non-canonical) isomorphism of triangles. Hence the following definition does not depend on auxiliary choices:

Definition

The chain complex $C_\bullet$ is called *$\mathfrak{I}$exact in degree n* if the composite map $X_n \xrightarrow{g_n} \Sigma C_n \xrightarrow{\Sigma f_{n+1}} \Sigma X_{n+1}$ belongs to $\mathfrak{I}$. It is called *$\mathfrak{I}$exact* if it is $\mathfrak{I}$exact in degree n for all $n \in \mathbb{Z}$.

If we recall from yesterday, in the diagram



the composition given by the stipled arrow is in $\mathfrak{I}$.

Lemma

Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor into some stable Abelian category $\mathfrak{C}$. Let $C_\bullet$ be a chain complex over $\mathfrak{T}$. The complex $C_\bullet$ is $\mathrm{Ker}F$ exact in degree n if and only if the sequence

$$F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1})$$

in $\mathfrak{C}$ is exact at $F(C_n)$.

Later we will meet homological ideals given as intersections kernels of a family of homological ideals.

Definition

An object $A \in\in \mathfrak{T}$ is called $\mathfrak{I}$*projective* if the functor $\mathfrak{T}(A, \sqcup) \colon \mathfrak{T} \to \mathfrak{Ab}$ is $\mathfrak{I}$exact.

We write $\mathcal{P}_{\mathfrak{I}}$ for the class of $\mathfrak{I}$projective objects in $\mathfrak{T}$.

Lemma

An object $A \in\in \mathfrak{T}$ is $\mathfrak{I}$projective if and only if $\mathfrak{I}(A, B) = 0$ for all $B \in\in \mathfrak{T}$. .

Lemma

The class $\mathcal{P}_{\mathfrak{I}}$ of $\mathfrak{I}$projective objects is closed under (de)suspensions, retracts, and possibly infinite direct sums (as far as they exist in $\mathfrak{T}$).

The following will supply us with projective objects.

Theorem

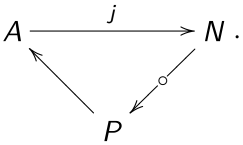
1. *Suppose that F is $\mathfrak{I}$-exact and $Q \in \mathfrak{T}$ satisfies*

$$\mathfrak{T}(Q, A) = \mathfrak{C}(X, F(A))$$

for some object X in $\mathfrak{C}$. Then Q is $\mathfrak{I}$-projective.

2. *Suppose that $\mathfrak{I} = Ker\ F$. Then an object P of $\mathfrak{T}$ is projective iff $F(P)$ is projective in $\mathfrak{C}$*

$\mathfrak{T}$ contains enough projectives if, for any object A in $\mathfrak{T}$, there exists an exact triangle of the form



with P projective and $j \in \mathfrak{I}(A, N)$. Then we can construct projective resolutions.

The phantom tower.

We will use above to construct a projective resolution.

A

The phantom tower.

We will use above to construct a projective resolution.

$$A = N_0$$

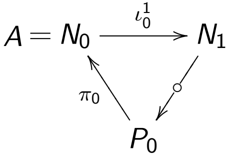
The phantom tower.

We will use above to construct a projective resolution.

$$\begin{array}{ccc} A = N_0 & & \\ & \nwarrow^{\pi_0} & \\ & & P_0 \end{array}$$

The phantom tower.

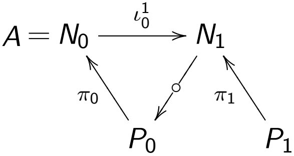
We will use above to construct a projective resolution.



mapping cone of π_0

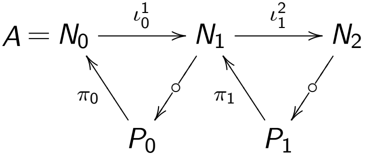
The phantom tower.

We will use above to construct a projective resolution.



The phantom tower.

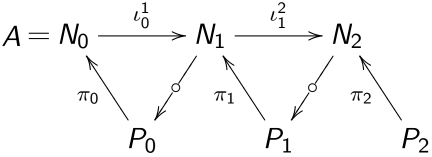
We will use above to construct a projective resolution.



mapping cone of π_1

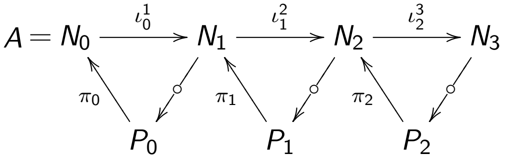
The phantom tower.

We will use above to construct a projective resolution.



The phantom tower.

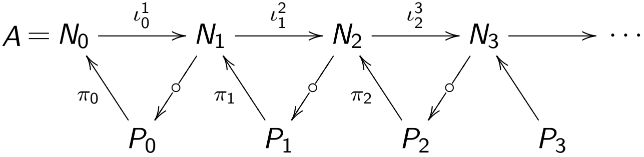
We will use above to construct a projective resolution.



mapping cone of π_2

The phantom tower.

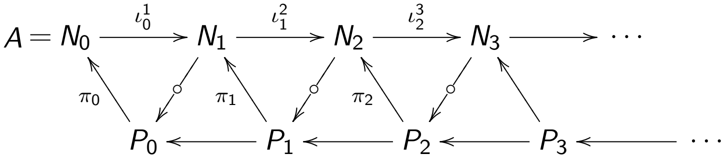
We will use above to construct a projective resolution.



etc.

The phantom tower.

We will use above to construct a projective resolution.



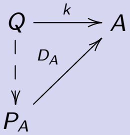
The succesive compositions produce the projective resolution of A

Example

1. $\mathfrak{T} = KK^{\Gamma}$ for a discrete group Γ
2. $j \in \mathfrak{J}$ if, for all torsion subgroups $H \subset \Gamma$, $j = 0$ in KK^{H}
3. $\mathcal{P}_{\mathfrak{J}}$ coincides with the usual class of proper Γ-algebras.

Definition

Given A, its projective cover is a $\mathfrak{J}$-projective object P_A and $D_A \in KK^{\Gamma}(P_A, A)$ such that every $k \in KK^{\Gamma}(Q, A)$ with $Q \in \mathcal{P}_{\mathfrak{J}}$ factorizes through D.



Example continued

Such a P_A always exists!

In fact, it is of the form $(P_{\mathbb{C}} \otimes A, D_{\mathbb{C}} \otimes 1)$, where $D_{\mathbb{C}}$ is the usual Dirac operator (mostly familiar if $B\Gamma$ is a spin-manifold)

γ element

Γ has a γ-element if $D_{\mathbb{C}}$ has the right inverse, i. e. there exists a $d \in KK^{\Gamma}(\mathbb{C}, P_{\mathbb{C}})$(dual Dirac) such that

$$d_{\mathbb{C}} \circ D_{\mathbb{C}} = id|_{P_{\mathbb{C}}}.$$

In this case the exterior Kasparov product with $\gamma = D_{\mathbb{C}} \circ d_{\mathbb{C}} \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$ is the projection onto the complement of $\mathcal{N}$, the $\mathfrak{I}$-contractible objects, and

$$KK^{\Gamma} =< \mathcal{P}_{\mathfrak{I}} > \oplus \mathcal{N}$$

Example continued

Given homological functor F, which vanishes on $\mathfrak{I}$, it has a total left derived functor $\mathbb{L}F$ with a natural transformation

$$\mathbb{L}F \to F.$$

In fact, $\mathbb{L}F(A) = F(P_A)$.

Theorem

Let $F(A) = K_*(A \rtimes_{red} \Gamma)$. Then $\mathbb{L}F(A) = K^*_\Gamma(A)$ and

$$K^*_\Gamma(A) = \mathbb{L}F(A) \to F(A) = K_*(A \rtimes_{red} \Gamma)$$

is the assembly map.

Example continued

Theorem

Suppose that Γ satisfies strong Baum-Connes conjecture, i. e. it has the γ-element equal to one. Then KK^Γ coincides with the localizing subcategory generated by $\mathcal{P}_{\mathfrak{J}}$.

Recall that f. ex. amenable groups satisfy the hypothesis. Moreover $\mathcal{P}_{\mathfrak{J}}$ is generated by homogeneous actions of Γ, hence any stable homological functor on KK^Γ which coincides with K_Γ^* on homogeneous actions is the same as K_Γ^*.

Corollary

Suppose that Γ satisfies the strong Baum-Connes conjecture, Γ acts on X and that A is a Γ-C^*-algebra in $\mathfrak{C}^*\mathfrak{alg}(X)$. If A is in $\mathcal{B}(X)$, then so is $A \rtimes_{red} \Gamma$.

Once we have a notion of exactness for chain complexes, we can do homological algebra in the homotopy category $\mathrm{Ho}(\mathfrak{T})$ of all chain complexes over $\mathfrak{T}$.

Definition

Let $\mathfrak{J}$ be a homological ideal in a triangulated category $\mathfrak{T}$ with enough projective objects. Let $F\colon \mathfrak{T} \to \mathfrak{C}$ be an additive functor with values in an Abelian category $\mathfrak{C}$. Applying F pointwise to chain complexes, we get an induced functor $\bar{F}\colon \mathsf{Ho}(\mathfrak{T}) \to \mathsf{Ho}(\mathfrak{C})$. Let $P\colon \mathfrak{T} \to \mathsf{Ho}(\mathfrak{T})$ be the functor that maps an object of $\mathfrak{T}$ to a projective resolution of $\mathfrak{T}$. Let $H_n\colon \mathsf{Ho}(\mathfrak{C}) \to \mathfrak{C}$ be the nth homology functor. The composite functor

$$\mathbb{L}_n F\colon \mathfrak{T} \xrightarrow{P} \mathsf{Ho}(\mathfrak{T}) \xrightarrow{\bar{F}} \mathsf{Ho}(\mathfrak{C}) \xrightarrow{H_n} \mathfrak{C}$$

for $n \in \mathbb{N}$ is called the nth *left derived functor* of F. If $F\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ is a contravariant additive functor, then the corresponding composite functor $\mathbb{R}^n F\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ is called the nth *right derived functor* of F.

More concretely, $\mathbb{L}_n F(A)$ for a covariant functor F and $A \in\in \mathfrak{T}$ is the homology of the chain complex

$$\cdots \to F(P_{n+1}) \xrightarrow{F(\delta_{n+1})} F(P_n) \xrightarrow{F(\delta_n)} F(P_{n-1}) \to \cdots \to F(P_0)$$

at $F(P_n)$ in degree n, where $(P_\bullet, \delta_\bullet)$ is an $\mathfrak{I}$projective resolution of A. Similarly, $\mathbb{R}^n F(A)$ for a contravariant functor F is the cohomology of the chain complex

$$\cdots \leftarrow F(P_{n+1}) \xleftarrow{F(\delta_{n+1})} F(P_n) \xleftarrow{F(\delta_n)} F(P_{n-1}) \leftarrow \cdots \leftarrow F(P_0)$$

at $F(P_n)$ in degree $-n$.

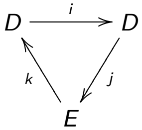
The phantom tower of A generates spectral sequences. For simplicity, we consider a homological functor $F\colon \mathfrak{T} \to \mathfrak{C}$ Since we are not going to use it later, we'll just describe the *exact couple* which generates it.

Let $N_n := A$ for $n \in -\mathbb{N}$ and define bigraded Abelian groups

$$D := \sum_{p,q\in\mathbb{Z}} D_{pq}, \qquad D_{pq} := F_{p+q+1}(N_{p+1}),$$

$$E := \sum_{p,q\in\mathbb{Z}} E_{pq}, \qquad E_{pq} := F_{p+q}(P_p),$$

and homogeneous group homomorphisms



$$\begin{aligned} i_{pq} &:= (\iota_{p+1}^{p+2})_* \colon D_{p,q} \to D_{p+1,q-1}, & \deg i &= (1,-1), \\ j_{pq} &:= (\varepsilon_p)_* \colon D_{p,q} \to E_{p,q}, & \deg j &= (0,0), \\ k_{pq} &:= (\pi_p)_* \colon E_{p,q} \to D_{p-1,q}, & \deg k &= (-1,0). \end{aligned}$$

Since F is homological, the chain complexes

$$\cdots \to F_{m+1}(N_{n+1}) \xrightarrow{\varepsilon_{n*}} F_m(P_n) \xrightarrow{\pi_{n*}} F_m(N_n) \xrightarrow{\iota_{n*}^{n+1}} F_m(N_{n+1}) \to \cdots$$

are exact for all $m \in \mathbb{Z}$. This means that (D, E, i, j, k) is an *exact couple*.

Theorem

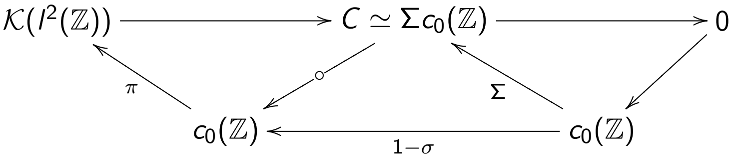
The ABC spectral sequence for homological functor is independent of auxiliary choices, functorial in Aand abuts at $F(A)$. The second tableaux involves only the derived functors:

$$E^2_{pq} \cong \mathbb{L}_p F_q(A),$$

There is a similar result for cohomological functors.

- $\Gamma = \mathbb{Z}$
- $\mathfrak{J} = Ker : \ KK^{\mathbb{Z}} \to \mathrm{KK}$

The $\mathfrak{J}$-projective resolution of $\mathbb{C}$ has the form



The projective cover of $\mathbb{C}$ is just the mapping cone

$$c_0(\mathbb{Z}) \to c_0(\mathbb{Z}) \to \Sigma C_{1-\sigma}.$$

But this is just the rotated exact triangle associated to the extension

$$0 \to \Sigma c_0(\mathbb{Z}) \to C_0(\mathbb{R}) \to c_0(\mathbb{Z}) \to 0,$$

the $*$-homomorphism $C_0(\mathbb{R}) \to c_0(\mathbb{Z})$ given by the evaluation $f \to f|_{\mathbb{Z}}$.

Conclusion

$P_{\mathbb{C}} = C_0(\mathbb{R}^2)$, $D = \overline{\partial}$, the usual Dirac operator (or rather its phase),

$$K^*_{\mathbb{Z}}(A) = K_*((A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z}) \to K_*(A \rtimes \mathbb{Z}),$$

where the assembly map is given by the product with Dirac operator.
The spectral sequence computing $K^*_{\mathbb{Z}}(A)$ becomes the six term exact sequence in K-theory associated to the extension

$$\Sigma(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \rightarrowtail (A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z} \twoheadrightarrow (A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z}$$

Since $A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \simeq A \otimes \mathcal{K}$, this is just the usual Pimsner-Voiculescu exact sequence.

We assume that G is a compact, connected group.

- $\mathfrak{T} = \mathrm{KK}^{\hat{G}}$, the KK-category of G-coalgebras,
- $\mathfrak{J} = Ker: \ KK^{\hat{G}} \to \mathrm{KK}$
- $F(A) = K^*(A \rtimes \hat{G})$

Theorem

Baum-Connes for coactions

$$< \mathcal{P}_{\mathfrak{J}} >= \mathrm{KK}^{\hat{G}}$$

The spectral sequence computing $K^*_{\hat{G}}(A) = \mathbb{L}F(A)$ becomes a spectral sequence for the K-theory of $B = A \rtimes \hat{G}$:

$$E^*_{p,q} \Longrightarrow K_{p+q}(A \rtimes \hat{G})$$

and the E^2-term has form

$$E^2_{-p,q} = H^p(R_G, K_q(A))$$

In terms of a G-algebra $B = A \rtimes \hat{G}$ we get

Corollary

Let G be a connected compact group and B a separable G- C^* algebra. Then

$$K^G_*(B) = K_*(B \rtimes G) = 0 \Longrightarrow K_*(B) = 0.$$