Let X be a topological space.

Definition

An algebra over a space X is a C*-algebra (always separable) A together with a continuous map

$$\phi : \mathrm{Prim}(A) \to X$$

In the case when X[1] is Hausdorff, this is the same as a $C(X)$-algebra, i. e. a $*$-homomorphism from $C(X)$ to the center of the multiplier algebra of A. Our topological spaces will be finite, always T_0 and we will use the arrows

$$1 \longrightarrow 2 \qquad \text{to denote specialisation relation: } 1 \in \overline{\{2\}}\ .$$

In other words, our spaces will come as finite partially ordered sets with Alexandrov topology, where the ordering is given by $\rightarrow$.
One more useful bit of notation. A basic open set is a subset of X of the form

$$U_x = \{y \in X \mid x \geq y\}.$$

[1] In general, the space X has to be sober, which means that every closed irreducible subset has a unique generic point.

In our case, the following simpleminded version will suffice.

Definition

An algebra over a space X is a C*-algebra A with a family of ideals $A(U)$ for each open subset of X, consistent with the inclusion order on open sets.

For a locally closed subset $Y = U \setminus V$ of X, with $V \subset U$, we set $A(Y) = A(U)/A(V)$. In particular, the fibers of A are given by $A_x = A(\{x\}) = A(U_x)/A(U_x \setminus \{x\})$.

Definition

A $*$-homomorphism

$$\phi : A \to B$$

of X-algebras is a $*$-homomorphism $\phi : A \to B$ which satisfies

$$\phi(A(U)) \subset B(U) \text{ for all open } U \subset X.$$

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For later reference, ϕ is said to be semisplit, if it has a completely positive splitting $\sigma : B \to A$ which satisfies

Definition

Let $\mathfrak{C}^*\mathfrak{alg}(X)$ be the category whose objects are the C^*-algebras over X and whose morphisms are the X-equivariant *-homomorphisms. We write $\mathrm{Hom}_X(A, B)$ for this set of morphisms.

Functoriality

Let X and Y be topological spaces. A continuous map $f: X \to Y$ induces a functor

$$f_*: \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y), \qquad (A, \psi) \mapsto (A, f \circ \psi).$$

.

In particular, the following are well defined

1. Induction. If $f\colon X \to Y$ is the embedding of a subset with the subspace topology, we also write

$$i_X^Y := f_*\colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y)$$

and call this the *extension* functor from X to Y.

2. Let X be a topological space and let Y be a *locally closed* subset of X, equipped with the subspace topology. Let (A, ψ) be a C^*-algebra over X. Its *restriction to* Y is a C^*-algebra $A|_Y$ over Y, consisting of the C^*-algebra $A(Y)$ equipped with the canonical map

$$\operatorname{Prim} A(Y) \simeq \psi^{-1}(Y) \xrightarrow{\psi} Y.$$

Adjointness properties

Let X be a topological space and let $Y \subseteq X$.

1. If Y is *open*, then there are natural isomorphisms

$$\mathrm{Hom}_X(i_Y^X(A), B) \cong \mathrm{Hom}_Y(A, r_X^Y(B))$$

if A and B are C^*-algebras over Y and X, respectively. In other words, i_Y^X is left adjoint to r_X^Y.

2. If Y is *closed*, then there are natural isomorphisms

$$\mathrm{Hom}_Y(r_X^Y(A), B) \cong \mathrm{Hom}_X(A, i_Y^X(B))$$

if A and B are C^*-algebras over X and Y, respectively. In other words, i_Y^X is right adjoint to r_X^Y.

3. For any locally closed subset $Y \subseteq X$, we have $r_X^Y \circ i_Y^X(A) = A$ for all C^*-algebras A over Y.

Example 1

$\underline{n} = \{1, 2, \ldots, n\}$, with topology given by the linear order

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n$$

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A C*-algebra over $\underline{n}$ is just a C*-algebra A with a distinguished linear chain of ideals

$$\{0\} = I_0 \lhd I_1 \lhd I_2 \lhd I_3 \lhd \cdots \lhd I_{n-1} \lhd I_n = A.$$

Where the ideals I_k are given by

$$I_k = A(\{1, 2, \ldots k\})$$

Example 2

A four point space $X = \{1, 2, 3, 4\}$ with topology given by

$$
\begin{tikzcd}
 & 1 \arrow[dl] \\
4 & 2 \arrow[l] \\
 & 3 \arrow[ul]
\end{tikzcd}
$$

Example 2

A four point space $X = \{1, 2, 3, 4\}$ with topology given by

$$
\begin{tikzcd}[row sep=small]
 & 1 \arrow[dl] \\
4 & 2 \arrow[l] \\
 & 3 \arrow[ul]
\end{tikzcd}
$$

A C*-algebra over X is the same as an extension of the form

$$A(\{4\}) \to A \to A(\{1\}) \oplus A(\{2\}) \oplus A(\{3\})$$

Bivariant Kasparov functor over X

To describe the cycles for $\mathrm{KK}_*(X; A, B)$ recall that the usual Kasparov cyles for $\mathrm{KK}_*(A, B)$

$$(\varphi, \mathcal{H}_B, F, \gamma) \text{ even case and } (\varphi, \mathcal{H}_B, F) \text{ in the odd case}$$

have the following structure:

- $\mathcal{H}_B$ is a right Hilbert B-module;
- $\varphi\colon A \to \mathrm{B}(\mathcal{H}_\mathrm{B})$ is a *-representation;
- $F \in \mathrm{B}(\mathcal{H}_\mathrm{B})$;
- $\varphi(a)(F^2 - 1)$, $\varphi(a)(F - F^*)$, and $[\varphi(a), F]$ are compact for all $a \in A$;
- in the even case, γ is a $\mathbb{Z}/2$-grading on $\mathcal{H}_B$.

Definition

Let A and B be C^*-algebras over X. A Kasparov cycle $(\varphi, \mathcal{H}_B, F, \gamma)$ or $(\varphi, \mathcal{H}_B, F)$ for $\mathrm{KK}_*(A, B)$ is called *X-equivariant* if

$$\varphi(A(U)) \cdot \mathcal{H}_B \subseteq \mathcal{H}_B \cdot B(U) \qquad \text{for all } U \in \mathbb{O}(X).$$

A homotopy is an X-equivariant cycle for $\mathrm{KK}(A, C([0,1]) \otimes B)$, where we view $C([0,1]) \otimes B$ as a C^*-algebra over X in the obvious way.

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Definition

$\mathrm{KK}_*(X; A, B)$ is the group of homotopy classes of X-equivariant Kasparov cycles for $\mathrm{KK}_*(A, B)$

Functoriality properties of $\mathrm{KK}_*(X; A, B)$

1. The groups $\mathrm{KK}_*(X; A, B)$ define a bifunctor from $\mathfrak{C}^*\mathfrak{sep}(X)$ to the category of $\mathbb{Z}/2$-graded Abelian groups, contravariant in the first and covariant in the second variable.
2. There is a natural, associative *Kasparov composition product*

$$\mathrm{KK}_i(X; A, B) \times \mathrm{KK}_j(X; B, C) \to \mathrm{KK}_{i+j}(X; A, C)$$

if A, B, C are C^*-algebras over X.
Furthermore, there is a natural *exterior product*

$$\mathrm{KK}_i(X; A, B) \times \mathrm{KK}_j(Y; C, D) \to \mathrm{KK}_{i+j}(X \times Y; A \otimes C, B \otimes D)$$

for two spaces X and Y and C^*-algebras A, B over X and C, D over Y

(All tensor products are minimal).

Properties:

1. Stability KK is invariant under $A \to A \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space.
2. Homotopy Given a continuous family of $*$-homomorphisms $t \to \phi_t : A \to C$, $\mathrm{KK}_1(X; \phi_t, \cdot)$ and $\mathrm{KK}_1(X; \cdot, \phi_t)$ are constant.
3. Bott periodicity: $A \to \Sigma A = C_0(\mathbb{R}, A)$ is an (odd)self-equivalence:
$$\mathrm{KK}_*(X; \Sigma A, B) \simeq \mathrm{KK}_*(X; A, \Sigma B) \simeq \mathrm{KK}_{*+1}(X; A, B).$$
4. The classes in $\mathrm{KK}_1(X; A, B)$ are given by semisplit over X extensions: $0 \to B \otimes \mathrm{K} \to E \to A \to 0$

5. Excision. Given a semisplit short exact sequence $0 \to I \to A \to A/I \to 0$, there exists an associated six term exact sequence

$$\begin{tikzcd} \mathrm{KK}_0(X; A/I, B) \arrow[r] & \mathrm{KK}_0(X; A, B) \arrow[r] & KK_0(X; I, B) \arrow[d, "\circ"] \\ KK_1(X; I, B) \arrow[u, "\circ"] & KK_1(X; A, B) \arrow[l] & KK_1(X; A/I, B) \arrow[l] \end{tikzcd}$$

and similarly in the second variable. For future reference we will write such 6-term exact sequences in the form of triangles

$$\begin{tikzcd} \mathrm{KK}_*(X; A/I, B) \arrow[rr] & & \mathrm{KK}_*(X; A, B) \arrow[dl] \\ & \mathrm{KK}_*(X; I, B) \arrow[ul, "\circ"] & \end{tikzcd}$$

Recall the (Kasparov) product

$$\mathrm{KK}_i(X; A, B) \times \mathrm{KK}_j(X; B, C) \to \mathrm{KK}_{i+j}(X; A, C)$$

It gives us the following:

Definition

$\mathfrak{KK}(X)$ is the category with objects given by C*-algebras over X and morphisms given by $\mathrm{KK}_0(X; \cdot, \cdot)$.

Mapping cone

Given a $*$-homomorphism $\phi : A \to B$ of X-C*-algebras, the mapping cone of ϕ is given by

$$C_\phi(U) = \{(f, a) \in C_0(]0,1], B(U)) \times A(U)) \mid f(1) = \phi(a)\}.$$

It fits into the exact sequence

$$0 \to B[-1] \to C_\phi \to A \to 0.$$

$(B[-1] = \Sigma B)$.
We will again write this exact sequence as a triangle

$$\begin{tikzcd} A \arrow[rr, "\phi"] & & B \arrow[dl, "\circ"] \\ & C_\phi \arrow[ul] & \end{tikzcd}$$

.

First some comments on triangulated categories. The notion was introduced to do homological algebra in non-abelian categories, where we do not have at our disposal kernels and cokernels of morphisms. Example is our category of C*-algebras over X, where there are no cokernels, or $\mathfrak{K}\mathfrak{K}(X)$, where the morphisms are given by abstract Kasparov cycles, hence even the notion of kernel makes no sense.

The idea is to replace both kernel and cokernel by a single object, the mapping cone.
The mapping cone exact sequence can be "unravelled" to give a long exact sequence of morphisms in $C^*(X)$

$$\ldots \to B[-2] \to C_\phi[-1] \to A[-1] \xrightarrow{\phi} B[-1] \to C_\phi \to A \xrightarrow{\phi} B \to ?$$

As it stands, it terminates on the right, but, if we go to $\mathfrak{K}\mathfrak{K}(X)$-category, we have at out disposal the Bott-periodicity equivalence $A \to \Sigma A$, hence we can use its inverse to continue to the right and get a two-sided infinite "exact" sequence in $\mathfrak{K}\mathfrak{K}(X)$:

$$\cdots \longrightarrow B[-2] \longrightarrow C_\phi[-1] \longrightarrow A[-1] \xrightarrow{\phi[-1]} B[-1] \longrightarrow C_\phi \longrightarrow A \xrightarrow{\phi} B$$

$$\longrightarrow C_\phi[1] \longrightarrow A[1] \xrightarrow{\phi[1]} B[1] \longrightarrow C_\phi[2] \longrightarrow A[2] \xrightarrow{\phi[2]} B[2] \longrightarrow \cdots$$

Hence the mapping cone triangle

$$\begin{tikzcd} A \arrow[rr, "\phi"] & & B \arrow[dl, "\circ"] \\ & C_\phi \arrow[ul] & \end{tikzcd}$$

represents the doubly infinite "exact" sequence above.

A triangulated category $\mathfrak{T}$ is an "abstraction" of above. It requires existence of

1. A *shift* functor
$$A \rightarrow \Sigma A$$
which is a self-equivalence of $\mathfrak{T}$
2. A class of distinguished triangles
$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl, "\circ"] \\ & C \arrow[ul] & \end{tikzcd}$$
written often as $\Sigma B \rightarrow C \rightarrow A \rightarrow B$

satisfying the following axioms.

Axioms

1. Any morphism $f\colon A \to B$ fits into a distinguished triangle (C-"mapping cone" of f)

$$\Sigma B \to C \to A \xrightarrow{f} B.$$

2. A triangle $\Sigma B \to C \to A \to B$ is distinguished if and only if $\Sigma A \to \Sigma B \to C \to A$ is distinguished.
3. Let $\Sigma B \to C \to A \to B$ and $\Sigma B' \to C' \to A' \to B'$ be distinguished triangles and let $f_A\colon A \to A'$, $f_B\colon B \to B'$ be morphisms in $\mathfrak{T}$ such that the compositions $A \to B \to B'$ and $A \to A' \to B'$ agree. One can fill in the following diagram

$$\begin{tikzcd} \Sigma B \arrow[r] \arrow[d, "\Sigma f_B"] & C \arrow[r] \arrow[d, dashed, "f_C"] & A \arrow[r] \arrow[d, "f_A"] & B \arrow[d, "f_B"] \\ \Sigma B' \arrow[r] & C' \arrow[r] & A' \arrow[r] & B'. \end{tikzcd}$$

4. The octahedral axiom. The diagram above can be filled in such a way that the mapping cone of the resulting morphism of triangles is again distinguished.

In the spirit of the discussion before, we'll use it to construct "exact" sequences. So, for example, given an sequence of morphisms,

$$A \to B \to C,$$

we will say that it is exact, if in the diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi"] & & B \arrow[dl, "\circ"] \arrow[rr, "\psi"] & & C \arrow[dl, "\circ"] \\
& C_\phi \arrow[ul] & & C_\psi \arrow[ul] \arrow[ll, dashed] &
\end{tikzcd}$$

the composition given by the stipled arrow is zero.

Theorem

$\mathfrak{KK}(X)$ has the structure of triangulated category

The shift functor $A \to A[-1]$

is the suspension

$$A[-1] = C_0(\mathbb{R}, A).$$

Given an extension $0 \to I \to A \xrightarrow{q} A/I \to 0$ we can construct a commutative diagram

$$\begin{tikzcd}
 & I \arrow[r] \arrow[d, "\iota"'] & A \arrow[r, "q"] \arrow[d, equal] & A/I . \arrow[d, equal] \\
A/I[-1] \arrow[r] & C_q \arrow[r] & A \arrow[r, "q"] & A/I
\end{tikzcd}$$

We will call the extension admissible, if ι is invertible in KK and, in this case, associate to it the exact triangle

$$\begin{tikzcd}
A/I \arrow[rr] & & \Sigma I \arrow[dl, "\circ"] \\
 & A \arrow[ul] &
\end{tikzcd}$$

Note that the rotated exact triangle (axiom 2) is given by

$$\begin{tikzcd}
A \arrow[rr] & & A/I . \arrow[dl, "\circ"] \\
 & C_q \arrow[ul] &
\end{tikzcd}$$

and excision shows that semisplit extensions are admissible.

The exact triangles

are the triangles KK-equivalent to the triangles of the form

$$\begin{tikzcd} A/I \arrow[rr] & & \Sigma I \arrow[dl, "\circ"] \\ & A \arrow[ul] & \end{tikzcd}$$

for admissible extensions $0 \to I \to A \xrightarrow{q} A/I \to 0$.

Definition

A localizing subcategory of $\mathfrak{T}$ generated by a given class of objects is the smallest triangulated subcategory containing given objects and closed under (countable)coproducts.

The point of this definition is, that one can define quotient category, by formally inverting morphisms whose mapping cones lie in the localizing subcategory.

Definition

We shall use the functors

$$P_Y := i_Y^X \circ r_X^Y : \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(X)$$

for $Y \in \mathbb{LC}(X)$. Thus $(P_Y A)(Z) \cong A(Y \cap Z)$ for all $Z \in \mathbb{LC}(X)$.

If $Y \in \mathbb{LC}(X)$, $U \in \mathbb{O}(Y)$, then we get an extension

$$P_U(A) \rightarrowtail P_Y(A) \twoheadrightarrow P_{Y \setminus U}(A) \tag{2.1}$$

in $\mathfrak{C}^*\mathfrak{alg}(X)$ because we have extensions $A(Z \cap U) \rightarrowtail A(Z \cap Y) \twoheadrightarrow A(Z \cap Y \setminus U)$ for all $Z \in \mathbb{LC}(X)$.

We recursively construct a canonical increasing filtration

$$\emptyset = \mathfrak{F}_0 X \subset \mathfrak{F}_1 X \subset \cdots \subset \mathfrak{F}_\ell X = X$$

of X by open subsets $\mathfrak{F}_j X$, such that the differences

$$X_j := \mathfrak{F}_j X \setminus \mathfrak{F}_{j-1} X$$

are discrete for all $j = 1, \ldots, \ell$. In each step, we let X_j be the subset of all open points in $X \setminus \mathfrak{F}_{j-1} X$—so that X_j is discrete—and put $\mathfrak{F}_j X = \mathfrak{F}_{j-1} X \cup X_j$. Equivalently, X_j consists of all points of $X \setminus \mathfrak{F}_{j-1} X$ that are maximal for the specialisation preorder $\prec$. Since X is finite, X_j is non-empty unless $\mathfrak{F}_{j-1} X = X$, and our recursion reaches X after finitely many steps.

Let A be a C^*-algebra over X. We equip A with the canonical increasing filtration by the ideals

$$\mathfrak{F}_j A := P_{\mathfrak{F}_j X}(A), \qquad j = 0, \ldots, \ell,$$

so that

$$\mathfrak{F}_j A(Y) = A(Y \cap \mathfrak{F}_j X) = A(Y) \cap A(\mathfrak{F}_j X) \qquad \text{for all } Y \in \mathbb{LC}(X). \tag{2.2}$$

Everything that follows is proved using the filtratio $\mathfrak{F}_j A$

Theorem

The following are equivalent for a separable C^*-algebra A over X:

1. *$A \in\in \mathfrak{KK}(X)$ belongs to the triangulated subcategory of $\mathfrak{KK}(X)$ generated by objects of the form $i_x(B)$ with $x \in X$, $B \in\in \mathfrak{KK}$.*
2. *$A \in\in \mathfrak{KK}(X)$ belongs to the localising subcategory of $\mathfrak{KK}(X)$ generated by objects of the form $i_x(B)$ with $x \in X$, $B \in\in \mathfrak{KK}$.*
3. *For any $Y \in \mathbb{LC}(X)$, $U \in \mathbb{O}(Y)$, the extension*

$$P_U(A) \rightarrowtail P_Y(A) \twoheadrightarrow P_{Y\setminus U}(A)$$

in $\mathfrak{C}^*\mathfrak{sep}(X)$ described above is admissible.

Furthermore, if A satisfies these conditions, then it already belongs to the triangulated subcategory of $\mathfrak{KK}(X)$ generated by $i_x(A(x))$ for $x \in X$.

Definition

Let $\mathfrak{KK}(X)_{\mathrm{loc}} \subseteq \mathfrak{KK}(X)$ be the full subcategory of all objects that satisfy above conditions.

Theorem

1. *Let X be a finite topological space. Let $A, B \in \mathfrak{KK}(X)_{\mathrm{loc}}$ and let $f \in \mathrm{KK}_*(X; A, B)$. If $f(x) \in \mathrm{KK}_*(A(x), B(x))$ is invertible for all $x \in X$, then f is invertible in $\mathfrak{KK}(X)$. In particular, if $A(x) \cong 0$ in $\mathfrak{KK}$ for all $x \in X$, then $A \cong 0$ in $\mathfrak{KK}(X)$.*
2. *Suppose that the extensions of C^*-algebras*

$$A(U_x \setminus \{x\}) \rightarrowtail A(U_x) \twoheadrightarrow A(x)$$

are semi-split for all $x \in X$. Then $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$. In particular, this applies if the underlying C^*-algebra of $A \in\in \mathfrak{KK}(X)$ is nuclear.

Definition

Let $\mathcal{B}(X)$ be the localizing subcategory of $\mathfrak{KK}(X)$ that is generated by $i_x(\mathbb{C})$ for $x \in X$.

Notice that $\{i_x(\mathbb{C}) \mid x \in X\}$ lists all possible ways to turn $\mathbb{C}$ into a C^*-algebra over X.

Theorem

Let X be a finite topological space and let $A \in\in \mathfrak{KK}(X)$. The following conditions are equivalent:

1. $A \in\in \mathcal{B}(X)$;
2. $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$ *and* $A(x) \in \mathcal{B}$ *for all* $x \in X$;

In addition, in this case $A(Y) \in\in \mathcal{B}$ *for all* $Y \in \mathbb{LC}(X)$.

Corollary

If the underlying C^*-algebra of A is nuclear, then $A \in \mathcal{B}(X)$ if and only if $A(x) \in \mathcal{B}$ for all $x \in X$.

Example

View a separable nuclear C^*-algebra A with only finitely many ideals as a C^*-algebra over $\mathrm{Prim}(A)$. The corollary above shows that A belongs to $\mathcal{B}(\mathrm{Prim}\,A)$ if and only if all its simple subquotients belong to the usual bootstrap class in $\mathfrak{KK}$.

Proposition

Let X be a finite topological space. Let $A, B \in \mathcal{B}(X)$ and let $f \in \mathrm{KK}_*(X; A, B)$. If f induces invertible maps $\mathrm{K}_*(A(x)) \to \mathrm{K}_*(B(x))$ for all $x \in X$, then f is invertible in $\mathfrak{KK}(X)$. In particular, if $\mathrm{K}_*(A(x)) = 0$ for all $x \in X$, then $A \cong 0$ in $\mathfrak{KK}(X)$.

Theorem

Let X be a finite topological space and let A be a separable C^*-algebra over X. The following are equivalent:

- *$A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$ and A_x is* KK*-equivalent to a nuclear C^*-algebra for each $x \in X$;*
- *A is* KK(X)*-equivalent to a C^*-algebra over X that is tight, separable, nuclear, purely infinite, and C^*-stable.*