Problem Session Copenhagen Aug 07

Homotopy theory & group actions workshop

(1) (Madsen) $PU(n) = U(n)/\text{Center} = U(n)/S^1$
$= SU(n)/\mathbb{Z}/n$

Calculate $H^*(BPU(n); \mathbb{Z}/p)$ $(p \mid n)$

Have fib $B\mathbb{Z}/n \to BSU(n) \to BPU(n)$
$BS^1 \to BU(n) \to BPU(n)$ } induced by $BU(n) \to BPU(n) \to K(\mathbb{Z}, 3)$

Rem: a) If $p = n$ then $H^*(BPU(p); \mathbb{Z}/p)$ known additively (Kameko–Yagita)

b) $n = 2 = p$ $SU(2) = S^3$ $PU(2) = SO(3)$.

(also appears in Bensons problem list problem 9.1)

Other refs: – Angelo Vistoli
– Vistoli-Vavpetič
Broto – Vistoli.

Rem: in $PU(p)$ two max elt ab subgps.

$b = \begin{pmatrix} 0 & 1 & 0 & .. \\ 0 & 0 & 1 & .. \\ 1 & 0 & . & 0 \end{pmatrix} \in C_p \subseteq \Sigma_p \subseteq U(p)$

$a = \begin{pmatrix} 1 & & \\ & \xi & \\ & & \ddots \; \xi^{p-1} \end{pmatrix}$ $[a, b] = c = \begin{pmatrix} \xi & & \\ & \ddots & \\ & & \xi \end{pmatrix}$

in Yagita paper

results: (A) $0 \to H^*(PU(p)) \hookrightarrow H^*(E_1)^{N_1} \oplus H^*(E_2)^{N_2} \to H^*(\langle a \rangle) \to$

Quillen's result $\to$ (B) $H^*(G) \xrightarrow[\text{F-iso}]{\simeq} \lim_{\leftarrow} H^*(E)$

(I. Madsen)

(2) $\Gamma_g = \pi_0(\mathrm{Diff}^+(F_g))$ F_g genus g surface.

Calculate a) $H^*(B\Gamma_g; \mathbb{Q})$

b) $H^*(BGL_g(\mathbb{Z}); \mathbb{Q})$

c) $H^*(BSp_g(\mathbb{Z}); \mathbb{Q})$

(known stably, unstably very hard, known $g \leq 4$ (approx))

(3) (Ádem) a) $H^*(BGL_n(\mathbb{F}_p), \mathbb{F}_p)$?

b) $H^*(BUT_n(\mathbb{F}_p); \mathbb{F}_p)$

Approach to UT_n.

$$UT_n(\mathbb{F}_2) \simeq (\mathbb{Z}/2)^{n-1} \rtimes UT_{n-1}$$

- E_2^{**} has the right Krull dim
- For $n \leq 4$ $E_2 = E_\infty$

  $n = 5$ maybe also ~~true~~

Question: Is this always the case?

(4) (Adem) "Conj": If G is a central extension

$$1 \to (\mathbb{Z}/2)^r \to G \to (\mathbb{Z}/2)^k \to 1.$$

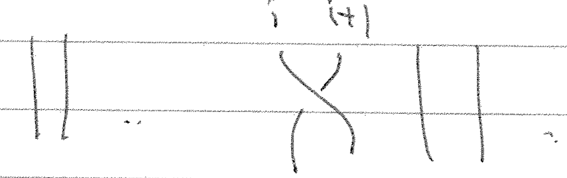
then the EMSS collapses at E_3.

Note: • Quillen calculated this for $r=1$, k even.

• Guggenheim-May calculate algebraic formulations for all differentials in terms of v_i

(5) (Westerland) $B_k =$ k'th Braid gp. $G =$ ft gp.

B_k acts on G^k (Hurwitz or Artin action).

$\sigma_i =$ 

$$\sigma_i(g_1, \dots, g_k) = (g_1, \dots, g_{i-1}, g_i, g_i g_{i+1} g_i^{-1}, \dots)$$

Pick $(g_1, \dots, g_k) =: \underline{g}$ $H_{\underline{g}} = \mathrm{Stab}\, \underline{g}$

What is $H^*(H_{\underline{g},k}; \mathbb{Q})$?

What is $\lim_{k\to\infty} H^*(H_{\underline{g},k}; \mathbb{Q})$?

Approach: Look at pure $1 \to PH_{\underline{g},k} \to H_{\underline{g},k} \to J \to 1.$
$J \subseteq \Sigma_k$

(G) X f.t. cpx

- How can we understand $BAut(X)$?

  - Is this like a "nice" grp (Lie, Arithmetic, p-compact, Kac-Moody. ...)

Rem a) Sullivan/Wilkerson $Out(X) = \pi_1 BAut(X)$ is an arithmetic grp.

b) $X = S^n$ $p=2$ $H^*(BAut(S^n); \mathbb{F}_2) \overset{\mathbb{F}\text{-iso}}{\simeq} H^*(BO(n+1); \ldots)$

$H^*(BAut(S^n); \mathbb{F}_p)$

- Does $BAut(X)$ have "max torus" and "Weyl grp"
  ~~ex~~ For $X = S^{n\wedge}_p$ is Weyl grp $\mathbb{Z}_p^\times \wr \Sigma_n$

- What is the Krull dim of $H^*(BAut(X))$

- What is the largest elt ab subgrp. Ie. largest k i.e. inj map $B(\mathbb{Z}/p)^k \longrightarrow BAut(X)$ & fin. H^* fg over steenrod alg.

- When is ~~~~ $[B\mathbb{Z}/2, BAut(X)]$ f.t. ??

Special cases $X = S^n \vee S^m$, $S^n \times S^m$

- Describe $map(BG, BAut(X))$

(5)

(Ambulation)

7 a) Does $Qd(p) = \mathbb{Z}/p \times \mathbb{Z}/p \rtimes SL_2(\mathbb{F}_p)$ act freely on a product of spheres? $(S^n \times S^m)$

- $\exists$ $\underset{\text{dim}}{\text{f.}}$ free G-CW-cpx
- geometric action?

b) Which f.t. gps G act freely on $S^n \times S^n$ (equi dimensional)

a) Does $G_p = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} * \in \mathbb{F}_p$ act freely on $S^{2pr-1} \times S^{2pr-1}$ for some (even?) $r \geq 1$.

(e.g. yes $r=1$ H-Unlu)

b) rank 2 p-gps — p odd? / $p=2$??

c) Classify free $\mathbb{Z}/p \times \mathbb{Z}/p$-actions on $S^n \times S^n$ (n odd)

- Does there always exist an equiv projections
- Can the actions with equiv proj be classified?

(8) (Hambleton) Alperin-Feng:
Is there a function $J: \mathbb{N} \to \mathbb{N}$.
s.t. for any f.t. grp acting smoothly + effectively on S
then $\exists\ A \trianglelefteq G$ abelian normal $[G:A] \leq J(n)$

Rem: Jordan proved this for linear actions in 1870's.

(9) (Hambleton/Pedersen) Classify f.t. Top grp actions on S^n (up to G-homeo / G-homotopy equiv) assuming each $(S^n)^H \subseteq S^n$ is a standardly embedded subsphere $\forall H \leq G$

Rem: PL-locally linear, $|G|$ odd done by Rothenberg-Madsen

(10) (Adem) If $(\mathbb{Z}/p)^r$ acts freely on $S^{n_1} \times \ldots \times S^{n_k}$ then $r \leq k$.

(11) (Adem) • If $\operatorname{rk} G = k$ then G acts freely on some $X \simeq S^{n_1} \times \ldots \times S^{n_k}$

• For $\operatorname{rank} G = 2$ p-grp $p \geq 3$ Smooth action will act on some $S^n \times S^m \iff Qd(p)$ not a subquotient

• All rank 2 simple grps act freely on some $S^n \times S^m$ except possibly $PSL_3(\mathbb{F}_p)$ or $Qd(p)$.

• $X = (S^1)^k$ $\quad 1 \to \mathbb{Z}^k \to \Gamma \to G \to 1$.

• Free and cocompact actions on $S^n \times \mathbb{R}^k$

12 [I. Madsen] Calculate

$$H^*(BS_\infty(\mathbb{Z}_p); \mathbb{F}_p) = ?$$

Rem: $H^*(BGL_\infty(\mathbb{Z}_p); \mathbb{F}_p)$

$\cong H^*(\mathrm{Im}\, J_p; \mathbb{F}_p) \otimes$

$H^*(B\,\mathrm{Im}\, J_p; \mathbb{F}_p) \otimes H^*(SU; \mathbb{F}_p)$

(Adem) (Bökstedt-M) (p odd)

(Connolly)

13 Conj $1 \to \mathbb{Z}^k \to \Gamma \to G \to 1$

$|G|$ odd, Γ periodic cohom.

$\Rightarrow$ Γ act freely & cocompactly on some $S^m \times \mathbb{R}^k$