

Def: (algebraic def) M a $\mathbb{Z}G$ -module

$$H^n(G; M) = \text{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, M)$$

$P_* \rightarrow \mathbb{Z}$ a proj resolution of $\mathbb{Z}$ (as trivial $\mathbb{Z}G$ -module).

(P_* satisfies $H_i(P_*) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{otherwise} \end{cases}$)

$$H^n(G; M) = H^n(\text{Hom}_{\mathbb{Z}G}(P_*, M))$$

(std. def from homological algebra)

Note: • Does not depend on resolution, i.e. choice of P_*

• Identifies with right derived functors assoc. to

$$M \mapsto M^G = H^0(G; M)$$

$$\textcircled{1} \quad G = \mathbb{Z}/N = \langle t \mid t^N = 1 \rangle \quad 1-t \in \mathbb{Z}G$$

$$N = 1+t+\dots+t^{N-1} \in \mathbb{Z}G$$

Have exact seq:

$$\dots \rightarrow \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{1+t+\dots+t^{N-1}} \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0$$

$g \mapsto 1$

Free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

$$\text{So } H^i(G; M) = \begin{cases} M^G & i=0 \\ M^G / (1+t+\dots+t^{N-1})M & i \text{ even} \\ \ker(1+\dots+t^{N-1}) / (1-t)M & i \text{ odd} \end{cases}$$

Note: groups are periodic in this case.

In particular
$$H^i(G; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/N & i \text{ even} > 0 \\ 0 & i \text{ odd} \end{cases}$$

② What about doing this for every gp?
(i.e. give an explicit res.)

Bar resolution $X_n = \text{free } \mathbb{Z}G\text{-module with basis } G^n$
 $[x_1, \dots, x_n] \quad n > 0$
 $[\]$

$$X_* \rightarrow \mathbb{Z} \quad [\] \mapsto 1.$$

$$\begin{aligned} d_n([x_1 | \dots | x_n]) &= x_1 [x_2 | \dots | x_n] \\ &+ \sum_{i=1}^{n-1} (-1)^i [x_1 | \dots | x_{i-1} | x_{i+1} | \dots | x_n] \\ &+ (-1)^n [x_1 | x_2 | \dots | x_n]. \end{aligned}$$

check this defines a free resolution

~~However~~, Give explicit ways to compute $H^*(G; M)$
 but this is unwieldy and impractical, but functorial in $($

Let $X \curvearrowright G$ as say G -CW-complex

and it is free (i.e. $\forall x \in X \{g \in G \mid gx = x\} = \{1\}$.)

$X \rightarrow X/G$ covering space $C_*(X)$ free $\mathbb{Z}G$ -chain complex

If X is a G -free and contractible then

$C_*(X) \rightarrow \mathbb{Z}$ is a free $\mathbb{Z}G$ -resolution.

Def: EG is a free G -space which is contractible.

$BG = EG/G$ is called the classifying space.

Properties: • unique up to G -homotopy

• $\forall$ G top sp contractible principal G -bundle

$$\begin{array}{c} G \rightarrow EG \\ \downarrow \\ BG \end{array}$$

"universal bundle" (Milnor).

no strictly $\frac{1}{2}g + \frac{1}{2}g$ to get etc.

Joins: $EG =$ formal $\mathbb{R}$ sums $\sum t_i g_i$

$t_i \geq 0$ for finitely many i 's.

$$\sum t_i = 1 \quad g_i \in G.$$

⚡ degree notation

gives weak topology making $t_i: EG \rightarrow [0,1]$.

$$\text{and } g_i = t_i^{-1}([0,1]) \rightarrow G$$

continuous

Has right G -action $(\sum t_i g_i, g) \mapsto \sum t_i g_i g$

This action is free (check).

Exercise: Understand what this means for $\mathbb{Z}/2$.

For any G top sp gives principal G -bundle.

$$\begin{array}{c} EG \\ \downarrow \\ BG = EG/G \end{array}$$

$$\text{Functorial } G \xrightarrow{f} K \rightsquigarrow \begin{array}{ccc} EG & \xrightarrow{Ef} & EK \\ \downarrow & & \downarrow \\ BG & \xrightarrow{Bf} & BK \end{array}$$

app $\rightarrow$ top.

$$B(G_1 \times G_2) = BG_1 \times BG_2 \quad (\text{exercise})$$

BG cell complex

$$C_*(EG) \rightarrow C_*(BG).$$

Examples:

① $G = \mathbb{Z}/2$ G acts freely on S^n via antipodal map $x \mapsto -x$.

$$S^n \subset S^{n+1} \subset \dots \subset S^\infty$$

$$EG = S^\infty, \quad BG = S^\infty / \mathbb{Z}_2 = \mathbb{R}P^\infty$$

New definitions: $H^*(G; R) = H^*(BG; R)$ as an algebra

R comm ring w. 1, trivial G -action.

G gives $\text{Ext}_{RG}^*(R, R)$.

Exercise: check this agrees with Yoneda Product

a product structure using algebraic topology.

② $G = \mathbb{Z}/p$ p odd $S^{2n+1} \rightarrow S^{2n+1}$ has a free action of S^1

$$\downarrow \\ \mathbb{C}P^n$$

$$S^{2n-1} \subset S^{2n+1} \subset \dots \subset S^\infty$$

infinite lens space.

$$BS^1 = \mathbb{C}P^\infty$$

$$B\mathbb{Z}/p = S^\infty / \mathbb{Z}/p = L_p^\infty.$$

Computations:

$$H^*(\mathbb{Z}/2; \mathbb{F}_2) = H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$$

$$\cong \mathbb{F}_2[e] \quad |e|=1$$

$$H^*(\mathbb{Z}/p; \mathbb{F}_p) \cong H^*(L_p^\infty; \mathbb{F}_p) = \Lambda(e) \otimes \mathbb{F}_p[b]$$

$$|e|=1 \quad |b|=2$$

$$\beta(e)=b.$$

Künneth formula:

$$H^*((\mathbb{Z}/p)^n; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[e_1, \dots, e_n] & \begin{matrix} |e_i|=1 \\ p=2 \end{matrix} \\ \Lambda(e_1, \dots, e_n) \otimes \mathbb{F}_p[b_1, \dots, b_n] & \begin{matrix} |e_i|=1 \quad |b_i|=2 \\ b_i = \beta(e_i) \quad p \text{ odd} \end{matrix} \end{cases}$$

Subgroups: $H \leq G$

$$\text{Have map } EG/H \rightarrow EG/G$$

$$\begin{array}{ccc} \pi & & \parallel \\ BH & \longrightarrow & BG \end{array}$$

$$\text{Induces map } H^*(BG; M) \longrightarrow H^*(BH; M|_H)$$

(here given with twisted coeff.)

$$\text{on cells } \text{Hom}_G(C_*(EG), M) \longrightarrow \text{Hom}_H(C_*(EG)_H, M|_H)$$

the restriction map.

(on H^0 just $M^G \rightarrow M^H$)

Have a map in the other direction

$$M^H \rightarrow M^G$$

$$x \mapsto \sum_{(g) \in G/H} gx$$

On Cohom we can likewise do this to get map

$$\text{Hom}(G \times (EG)_{|H}, M_{|H})^H \rightarrow \text{Hom}(G(EG), M)^G$$

$$H^*(H; M_{|H}) \rightarrow H^*(G; M)$$

the transfer

Properties:

$$\textcircled{1} \quad \text{tr}_{H \rightarrow G}^G : H^*(G; M) \rightarrow H^*(G; M)$$

is mult by the index $[G:H]$.

$$\left(\begin{array}{l} \text{generalizes } M^G \rightarrow M^H \xrightarrow{\text{tr}} M^G \\ x \mapsto x \mapsto [G:H]x \end{array} \right)$$

Application: $|G| \cdot H^i(G; M) = 0$ for $i > 0$

since

$$\begin{array}{ccc} H^i(G; M) & \xrightarrow{\text{res}} & H^i(H; M) \\ |G| \searrow & & \swarrow \text{tr} \\ & H^i(G; M) & \end{array} \quad \sim \begin{cases} M & i=0 \\ 0 & i>0 \end{cases}$$

• If $|G| \nmid \dim M \rightarrow M$ is an iso then

$$H^i(G; M) = 0 \text{ for } i > 0.$$

②

$$\text{Hom}_H(C_*(EG), M) \rightarrow \text{Hom}_{gHg^{-1}}(C_*(EG), M)$$

$$c_g: M \longrightarrow gHg^{-1}$$

induces

$$c_g^*: H^*(H; M) \rightarrow H^*(gHg^{-1}; M)$$

twisted coeff by c_g

(*)

$$\text{res}_K^G \text{tr}_H^G x = \sum_{i=1}^n \text{res}_{K \cap g_i H}^{K} \text{res}_{K \cap g_i H}^{g_i H} c_{g_i}^* x$$

$$G = \coprod_{i=1}^n K g_i H$$

double coset decomposition

One more application:

If $\text{Syl}_p(G) \leq H$ then the restriction map

$$H^*(G; \mathbb{F}_p) \rightarrow H^*(H; \mathbb{F}_p)$$

is a monomorphism

and its image can be explicitly determined using (*).

Assumption

$$H, K \subseteq G$$

$$\text{res}_H^G: H^*(G; M) \rightarrow H^*(H, M|_H)$$

$$\text{tr}_H^G: H^*(H; M|_H) \rightarrow H^*(G; M)$$

Basic properties:

$$\textcircled{1} \text{tr}_H^G \text{res}_H^G x = [G:H] x$$

$$\textcircled{2} \text{res}_K^G \text{tr}_H^G = \sum_{i=1}^n \text{tr}_{K \cap g_i H}^K \text{res}_{K \cap g_i H}^{g_i H} c_{g_i}$$

$$G = \coprod_{i=1}^n K g_i H \quad (\text{Cartan-Eilenberg})$$

Conseq: $H^*(G; \mathbb{F}_p) \xleftrightarrow{\text{res}} H^*(H; \mathbb{F}_p)$

What is the image: "stable elements."

$$P = \text{Stab}_P(G)$$

$$G = \coprod_{i=1}^n P g_i P$$

is maximal w.r.t. $\mathbb{F}_p$.

$$\begin{array}{ccc} H^*(G) & \rightarrow & H^*(P) \\ & \searrow \text{res} & \uparrow \text{res} \\ & & H^*(g_i P \cap P) \end{array}$$

$$x \in \text{im res}_P^G \iff \text{res}_{g_i P \cap P}^P c_{g_i}^* x = \text{res}_{g_i P \cap P}^P x.$$

Special case: $P \trianglelefteq G$ $g_i P = P$

Condition becomes

$$c_{g_i}^* x = x$$

$$H^*(G) \xhookrightarrow{\sim} H^*(P)^{G/P}$$

Exercise: Check that G acts trivially on $H^*(G)$ (we've used this!)

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1 \quad \text{gp ext.}$$

How does $H^*(H)$ and $H^*(K)$ relate to $H^*(G)$? _____

EG free contractible G -space

$$\begin{array}{ccc} EG/H \cong BH & \text{so an} & EG/H \cong BH \\ \downarrow & \text{fibration} & \downarrow \\ G \quad (EK \times EG/H)/K \cong BG & & \\ K & & \downarrow \\ & & BK \end{array}$$

The same spectral sequence has the form

$$E_2^{p,q} = H^p(K; H^q(H; M)) \Rightarrow H^{p+q}(G; M)$$

Lyndon-Hochschild-Serre spectral sequence.

Example H = Sylow p -subgp of G , $H \triangleleft G$.

$$E_2^{p,q} = H^p(K, H^q(H, \mathbb{F}_p)) \quad \text{order of } K \text{ prime to } p!$$

$$= \begin{cases} H^q(H, \mathbb{F}_p)^K & p=0 \\ 0 & p>0. \end{cases}$$

This is our example from before w stable elements formula.

Example: $G = A_4$. 2-Sylow = $\mathbb{Z}/2 \times \mathbb{Z}/2 =: V$ is normal

$\mathbb{Z}/3$ acts on V via sgs $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_2)$

$$H^*(V; \mathbb{F}_2) \cong \mathbb{F}_2[e_1, e_2] \quad |e_1| = |e_2| = 1.$$

this identifies with $S(V^*)$ symmetric on the dual V^*

$$H^*(A_4; \mathbb{F}_2) \cong \mathbb{F}_2[e_1, e_2]^{\mathbb{Z}/3} = \mathbb{F}_2[u_2, v_3, w_3] / u_2^3 + v_3^2 + w_3^2 + v_3 w_3$$

extend to $\mathbb{F}_4$ and diagonalize.

~~Example~~ At $p=3$

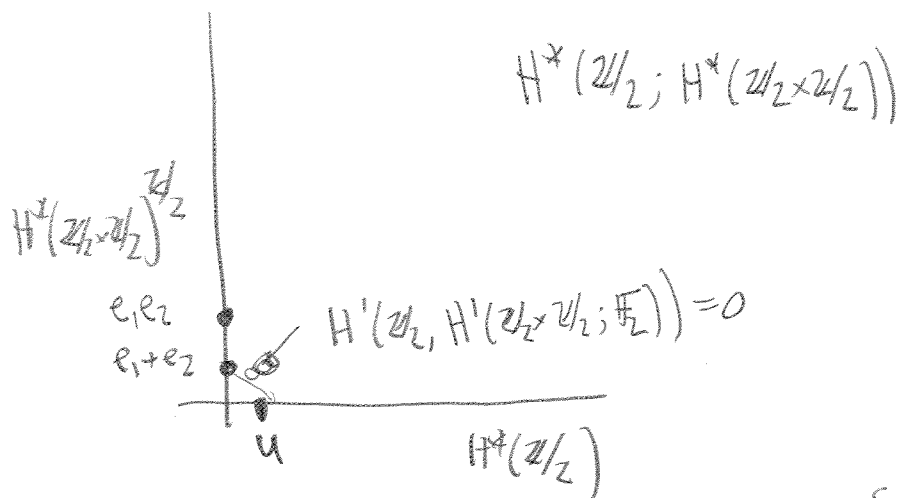
$$H^*(A_4; \mathbb{F}_3) \cong H^*(\mathbb{Z}/3; \mathbb{F}_2) \quad \text{by LHSSS.}$$

Example: D_8 dihedral of order 8.

$$1 \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow D_8 \rightarrow \mathbb{Z}/2 \rightarrow 1 \quad (D_8: \mathbb{Z}/2 \wr \mathbb{Z}/2)$$

$$E_2^{p,q} = H^p(\mathbb{Z}/2; H^q(\mathbb{Z}/2 \times \mathbb{Z}/2))$$

$$H^*(\mathbb{Z}/2 \times \mathbb{Z}/2)^{\mathbb{Z}/2} \cong \mathbb{F}_2[e_1, e_2]^{\mathbb{Z}/2} = \mathbb{F}_2[e_1 + e_2, e_1, e_2].$$



Since semidirect product, there's a section.

So no differentials to base

So $e_1 + e_2 \in E_{\infty}^{0,1}$ permanent cycle.

$$H^*(\mathbb{Z}/2; H^*(\mathbb{Z}/2 \times \mathbb{Z}/2)) \cong \tilde{H}_2[u] \otimes \mathbb{F}_2[e_1, e_2, e_1 e_2] / u(e_1 + e_2)$$

$$\oplus \mathbb{F}_2[e_1 + e_2, e_1 e_2].$$

$$\text{So: } E_2^{*,*} = \mathbb{F}_2[e_1 + e_2, u, e_1 e_2] / u(e_1 + e_2)$$

Also check no differentials

$$E_{\infty}^{*,*}$$

Check no extension problems so $E_{\infty}^{*,*} = H^*(D_8)$.

General structure of $H^*(G, \mathbb{F}_2)$.

Basic fact $\exists \rho: G \hookrightarrow U(n)$.

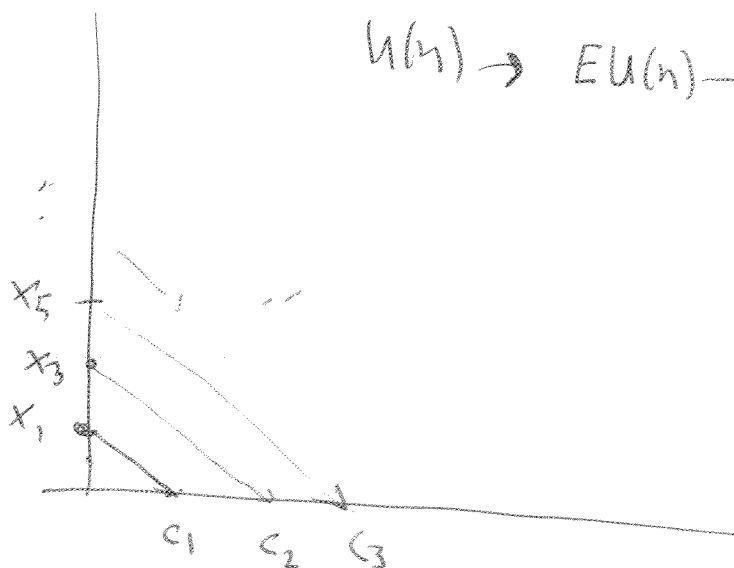
so: G acts freely on $U(n)$ by transitive

$$\begin{array}{ccccccc} G & \rightarrow & U(n) & \rightarrow & U(n)/G & \rightarrow & BG \rightarrow BU(n) \\ & & \text{fib.} & & \text{fib.} & & \text{fib.} \\ & & & & EG \times_G U(n) & & \end{array}$$

$$H^*(U(n); \mathbb{Z}) = \Lambda(x_1, x_3, \dots, x_{2n-1}). \quad |x_i| = i.$$

$$H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n] \quad |c_i| = 2i$$

$$U(n) \rightarrow EU(n) \rightarrow BU(n).$$



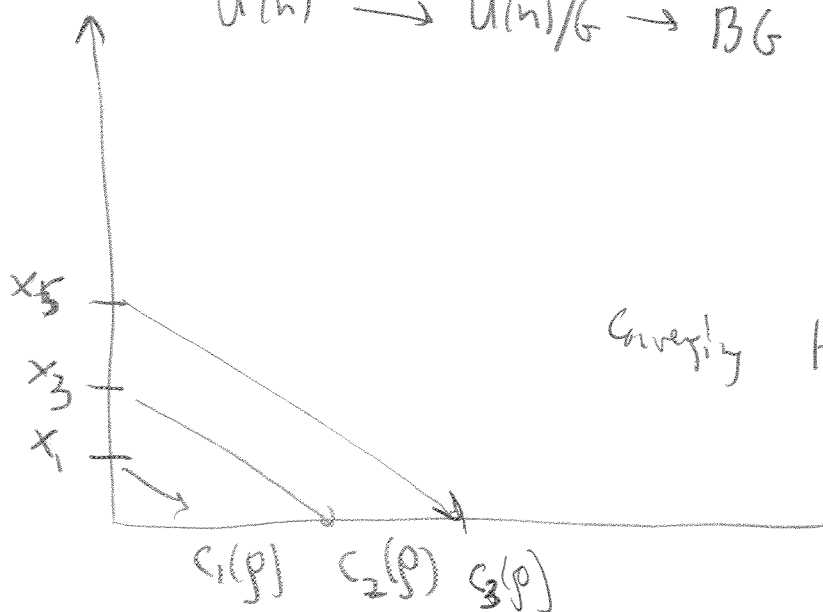
Here comparison $U(n) \rightarrow U(n)/U(n) \rightarrow BU(n)$

$$\uparrow$$

$$U(n) \rightarrow U(n)/G \rightarrow BG.$$

So by s.s. comparison

$$U(n) \rightarrow U(n)/G \rightarrow BG$$



Converging to $H^*(U(n)/G)$

Consequence: $H^*(G)$ is a finitely generated module over $H^*(BU(n)) = \mathbb{Z}[c_1, \dots, c_n]$
(Venkov)

Def:
$$P_G(t) = \sum_{i=0}^{\infty} \dim H^i(G; \mathbb{F}_p) t^i$$

Poincaré series.

Ex:
$$P_{\mathbb{Z}/2}(t) = 1 + t + t^2 + \dots = \frac{1}{1-t}$$

$$P_{(\mathbb{Z}/2)^n}(t) = \frac{1}{(1-t)^n}$$

$$P_{D_8}(t) = \frac{1}{(1-t)^n}$$

Conseq of theorems of Venkov.

$$P_G(t) = \frac{r(t)}{\prod_{i=1}^m (1-t^{2i})}$$

Note: pole at $t=1$!

Thm (Quillen (the Atiyah-Sum con))
Venkov.

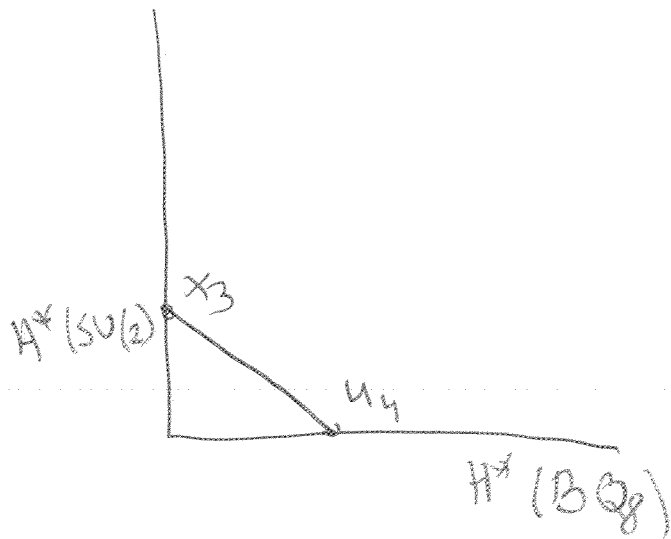
Order of the pole for $P_G(t)$ at $t=1$

is $r_p(G) = \max \{r \mid (\mathbb{Z}/p)^r \subseteq G\}$.

Ex: $G = Q_8 \hookrightarrow SU(2)$ $BQ_8 \rightarrow BSU(2)$.

$$H^*(BSU(2)) = \mathbb{F}_2[u_4]$$

$$SU(2) \rightarrow SU(2)/Q_8 \rightarrow BQ_8 \rightarrow BSU(2)$$



$$H^*(SU(2)/Q_8) = \mathbb{F}_2[x_1, \cancel{y_1}] / \begin{matrix} x_1^2 + x_1 y_1 + y_1^2 \\ x_1^2 y_1 + x_1 y_1^2 \end{matrix}$$

$H^*(BQ_8)$ free over $\mathbb{F}_2[u_4]$ with basis $H^*(SU(2)/Q_8)$.

$$H^*(Q_8) = \mathbb{F}_2[x_1, y_1, u_4] / \begin{matrix} x_1^2 + x_1 y_1 + y_1^2 \\ x_1 y_1^2 + x_1^2 y_1 \end{matrix}$$

Minimal resolutions & Gp cohomology

$$M \longleftrightarrow H^*(G; M)$$

G f.g. gp. P f.g. p.gp.

M f.g. KP -mod. K field of char p (e.g. $\mathbb{F}_p$)
 P p -gp.

$$F \twoheadrightarrow M \rightarrow 0$$

F free (projective)
 of minimal dim over K (proj. gen. of M)
 mapping onto M

$$I = \ker(KP \rightarrow K)$$

$$M \rightarrow M/I_M \rightarrow 0$$

H_0^1 by version of Nakayama's lemma.

by choosing F minimal is the same thing as
 choosing one copy of KP for each copy of KP .

$$\text{i.e. } \dim_K F = |P| \cdot \dim_K (M/I_M)$$

$$\dim_K F = |P| \cdot \dim_K H^0(P, M^*)$$

$$\text{Define } \Omega^1(M) = \ker(F \twoheadrightarrow M)$$

Continuing this way we get resolution

$$\begin{array}{ccccccc} \cdots & \rightarrow & F_2 & \rightarrow & F_1 & \rightarrow & F_0 \rightarrow M \\ & & \downarrow & \nearrow & \downarrow & \nearrow & \\ & & \Omega^2(M) & & \Omega^1(M) & & \end{array}$$

$$\begin{aligned} \dim F_1 &= |P| \cdot \dim H^0(P; \Omega^1 M^*) \\ &= |P| \cdot H^1(P, M^*) \end{aligned}$$

In general $\dim_k F_i = |P| \cdot \dim_k H^i(P; M^*)$
for a minimal resolution.

We have see: 1) M projective $\iff$

$$2) H^i(P; M) = 0 \text{ for } i > 0.$$

$\iff$

$$3) H^i(P; M) = 0 \text{ some } i > 0.$$

∇ : Downward implications clear. $3) \Rightarrow 1)$ follows from dim formula above together with the fact that if resolution stops we can split it since $kP^* \cong kP$. NB

Thm (Serre) Let P be a ft. p -gp which is not elementary abelian then there exists no non-zero classes $\alpha_1, \dots, \alpha_n \in H^1(P; \mathbb{F}_p)$ s.t. $\beta(\alpha_1)\beta(\alpha_2)\dots\beta(\alpha_n) = 0$.

$$\text{where } \beta: H^1(P; \mathbb{F}_p) \rightarrow H^2(P; \mathbb{F}_p)$$

induced by
left $\mathbb{Z}/p \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p$

$$\begin{aligned} H^1(P; \mathbb{F}_p) \\ \parallel \\ \text{Hom}(P, \mathbb{Z}/p) \end{aligned}$$

$$\begin{aligned} H^1(\mathbb{Z}/p, \mathbb{F}_p) \ni \text{identity} \\ \parallel \\ \text{Hom}(\mathbb{Z}/p, \mathbb{F}_p) \end{aligned}$$

$$\beta(\text{id}) = \beta(e) \in \text{Ext}_{\mathbb{F}_p[\mathbb{Z}/p]}^2(\mathbb{F}_p, \mathbb{F}_p)$$

represented by

$$\begin{array}{ccccccc} \mathbb{F}_p & \rightarrow & \mathbb{F}_p[\mathbb{Z}/p] & \rightarrow & \mathbb{F}_p[\mathbb{Z}/p] & \rightarrow & \mathbb{F}_p \rightarrow 0 \\ & & & \searrow & \nearrow & & \\ & & & I & & & \end{array}$$

recall: $p=2$ $H^*(\mathbb{Z}/p) = \Lambda(e) \otimes \mathbb{F}_p[e]$
 $p=2$ $H^*(\mathbb{Z}/2) = \mathbb{F}_2[e] \quad (pe = e^2)$

$$H^1(P; \mathbb{F}_p)$$

$$\begin{array}{ccccccc} 0 & \rightarrow & H & \rightarrow & P & \xrightarrow{\alpha} & \mathbb{Z}/p \rightarrow 1 \\ & & \uparrow & & & & \\ & & \text{maximal} & & & & \end{array}$$

So we see $\beta(\alpha)$ represented by

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{F}_p[P/H] \xrightarrow{1-t} \mathbb{F}_p[P/H] \rightarrow \mathbb{F}_p \rightarrow 0.$$

Ex. $H_1, \dots, H_n$ kernels assoc to $\alpha_1, \dots, \alpha_n$

Get $\mathbb{F}_p \rightarrow \mathbb{F}_p[G/H_n] \rightarrow \mathbb{F}_p[G/H_n] \rightarrow \mathbb{F}_p[G/H_{n-1}] \rightarrow \mathbb{F}_p[G/H_{n-1}] \rightarrow \mathbb{F}_p[G/H_{n-2}] \rightarrow \dots \rightarrow \mathbb{F}_p[G/H_1] \rightarrow \mathbb{F}_p$

rep. elt in $\text{Ext}_{\mathbb{F}_p}^2(\mathbb{F}_2, \mathbb{F}_1)$

Which represents $\beta(x_1) \dots \beta(x_n)$

We are assuming that classes chosen as
series then guarantees you can so

$$\beta(x_1) \dots \beta(x_n) = 0.$$

By truncation cpx above get C_x

$$C_{2i-1} = C_{2i-2} = \mathbb{F}_p[G/H_i] \quad i=1, \dots, n$$

$$H_\ell(C_x) = \begin{cases} \mathbb{F}_p & \ell = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Assume M f.g. kP -module.

Prop: M is projective $\Leftrightarrow M/H$ is projective
 $\forall$ maximal subgrp H .

proof: Consider $C^* \otimes M$
want to compute $H^*(P; C^* \otimes M)$
"hypercohomology w coeff in $C^* \otimes M$ ".

Using two different spectral sequences.

$$\textcircled{1} E_2^{r,q} = H^r(P; H^q(C^* \otimes M)) \Rightarrow H^{r+q}(P; C^* \otimes M)$$

$$\text{get seq } H^{l-2n}(P; M) \xrightarrow{d} H^l(P; M) \rightarrow H^l(P; C^* \otimes M) \rightarrow \dots$$

Since kp by $M \otimes P = 0$ by 1.3.3.1. $\rightarrow H^{l-2n+1}(P; M) \rightarrow \dots$

Hence $H^r(P; C^* \otimes M) =$

$$H^r(P; M) \oplus H^{r-2n+1}(P; M)$$

$$(2) \quad E_1^{r,q} = H^r(P, C^r \otimes M) \Rightarrow H^{r+q}(P; C^* \otimes M)$$

$$H^q(P; C^r \otimes M) \supset H^q(P; \mathbb{F}_0[P/H_i] \otimes M) \\ = H^q(H_i, M)$$

We assume M_H proj $\forall H$ maximal

$$\text{So } \forall q > 0 \quad H^q(P, C^r \otimes M) = 0.$$

and likewise $C^r = 0$ for $r > 2n+1$

Hence ~~by~~ by comparing the two s.s.

Hence for t sufficiently large

$$H^t(P; C^* \otimes M) = 0$$

By comparing the 2 s.s.

$$\text{we see } H^t(P; M) = 0 \text{ for } t \gg 0.$$

So P is projective

□

We get the following as cor:

Thm (Chavínad)

For any f.g. G , a kG -module M
is projective $\Leftrightarrow M|_E$ is proj $\forall$ abelian
 P -subgrps E of G .

Proof: Let $P = \text{Syl}_p(G)$

If $M|_P$ is free, then so is $kG \otimes_P M$
and this contains M as a direct summand, so
enough to check on P -grps.

Apply proposition repeatedly...

V_* a graded k -vector space.

Define the "growth rate"

$$\gamma(V_*) = \min \{ n \geq 0 \mid \lim_{t \rightarrow \infty} \frac{\dim_k V_t}{t^n} \neq 0 \}$$

(Algebra-Evens)

Def: Let M be a f.g. kG -module $P_* \rightarrow M$ $\leftarrow$ minimal resolution.

Define the complexity $cx_G(M)$ of M is

$$cx_G(M) = \gamma(P_*)$$

Example: $M = k[x_1, \dots, x_n]$

Thm (Quillen/Venkov, Alparin-Evens)

$$cx_G(M) = \max_{\substack{E \subseteq G \\ E \text{ effal}}} \{ cx_E(M|_E) \}.$$

pf: M is a direct summand of $M|_P \oplus_P kP$
hence $cx_G(M) = cx_P(M|_P)$

$$\text{But } cx_P(M|_P) = \delta(H^*(P; M^*))$$

proving about
min res
we see

$$= \max_E \delta(H^*(E; M^*))$$

$$\begin{array}{l} \text{S.S used} \\ \text{in pf of prop} \end{array} = \max_E cx_E(M). \quad \begin{array}{l} \text{min} \\ \text{result.} \end{array}$$

Cor: for $M = k$ trivial module

$$\begin{aligned} cx_P(k) &= \max_{E \subseteq P} cx_E(k) = \max_{E \subseteq P} \delta(H^*(E)) \\ &= r_P(P) := \max \{ r \mid (\mathbb{Z}/P)^r \leq P \}. \end{aligned}$$

i.e. rank = Krull dimension.

Lecture 4 Computations & further structure in group cohomology

$$\begin{array}{ccc} H^*(G; \mathbb{F}_p) & \xrightarrow{\text{res}_E^G} & H^*(E; \mathbb{F}_p) \\ & \searrow \text{res}_P^G & \nearrow \text{res}_E^P \\ & H^*(P; \mathbb{F}_p) & \end{array}$$

Def: Let $\mathcal{F}$ be a family of subgroups i.e.

- ① $H \in \mathcal{F} \implies H' \leq H \implies H' \in \mathcal{F}$
- ② $H \notin \mathcal{F} \implies g \in G \implies {}^g H \in \mathcal{F}$

Notation $H^*(\cdot) := H^*(\cdot; \mathbb{F}_p)$

$$H \in \mathcal{F} \quad H' \leq H \quad H^*(H) \xrightarrow{\text{res}_{H'}^H} H^*(H')$$

$$K \in \mathcal{F} \quad {}^g K \in \mathcal{F} \quad H^*(K) \xrightarrow[\cong]{c_{g^{-1}}^*} H^*({}^g K)$$

$$\begin{array}{ccc} H^*(G) & \longrightarrow & H^*(H) \\ \alpha \downarrow & \searrow \text{res}_H^G(\alpha) & \downarrow \\ & & H^*(H') \\ & \searrow & \downarrow \text{res}_{H'}^H \\ & & H^*(H') \\ & \searrow & \downarrow \text{res}_{H'}^G \\ & & H^*(H') \end{array}$$

$$\lim_{H \in \mathcal{F}} H^*(H) :=$$

squares $\{\alpha_H\}_{H \in \mathcal{F}}$ s.t.

$$\textcircled{1} \quad H' \subset H \quad \alpha_{H'} = \text{res}_{H'}^H(\alpha_H)$$

$$\textcircled{2} \quad K = {}^g H \quad \alpha_K = c_{g^{-1}}^*(\alpha_H)$$

We have map

$$H^*(G) \xrightarrow{q_F} \lim_{H \in F} H^*(H).$$

Thm (Cartan-Eilenberg)

Let $F = S_p^*(G)$ the family of all p -subgroups in G

then

$$H^*(G) \xrightarrow{q_{S_p}} \lim_{P \in F} H^*(P)$$

is an isomorphism.

Thm (Quillen) Let $E A_p^*(G)$ denote the family of ~~the~~ elementary abelian p -subgroups in G

then $H^*(G) \xrightarrow{q} \lim_{E \in F} H^*(E)$ is an isomorphism

ie. (1) kernel of q is nilpot

(2) if $u \in \lim_{E \in F} H^*(E)$ then for $\forall n \gg 0$ ^{some} $u^n \in \text{Im}(q_{A_p^*})$

Def: A collection of subgrp $H_1, \dots, H_n$ in G detects $H^*(G)$

if $H^*(G) \rightarrow \bigoplus_{i=1}^n H^*(H_i)$

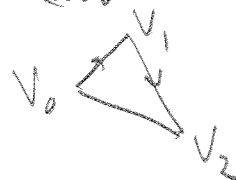
$\alpha \mapsto (\alpha|_{H_1}, \dots, \alpha|_{H_n})$ is inj.

Computational methods:

① By Cartan-Eilenberg can reduce the calculation of $H^*(G)$ to calculating $H^*(P)$ $P \in \text{Syl}_p(G)$ and understanding the stable elements.

② Determine the cohomology up to nilpotence.
 - describe $A_p(G)$ (nontrivial subgroups),
 - calculate $H^*(G) \rightarrow H^*(E)^{N_G E}$ and piece together.

③ $A_p(G)$ as a partially ordered set under inclusion.
 $V_0 < V_1 < \dots < V_n$ ~~n~~-simplex.



introduced by Brown + Quillen.

G acts by conjugation

$$E G \times_G |A_p(G)| \xrightarrow{\text{mod } p \text{ coh iso if } p \nmid |G|} BG$$

$$\downarrow$$

$$|A_p(G)|/G$$

isotropy subgroups are intersections of normalizers of elementary abelian subgroups.

There is a spectral sequence with

$$\text{isotropy} \quad E_{i,7}^{r,7} = H^7(G; C^r(|A_p(G)|)) \Rightarrow H_G^{r+7}(|A_p(G)|)$$

Theorem: (Webb) The spectral sequence collapses at E_2 and

$$E_2^{p,q} = \begin{cases} H^q(G) & p=0 \\ 0 & p>0. \end{cases}$$

Cor: $H^*(G)$ can be computed from $\{H^*(G_\sigma)\}$ Is from the chemistry of interactions of normalizers of p -subgr in G .

Rem: If G f.t. gp of Lie type in char p then $S_r(G) \xrightarrow{C} \text{ Tits building.}$

Ex: ① Symmetric groups $H^*(\Sigma_\infty)$ is known by Nakaoaka et

$$H^*(\Sigma_\infty) \rightarrow H^*(\Sigma_n)$$

$p=2$ $H^*(\Sigma_n) \simeq \lim_{E \in A_2(G)} H^*(E)$

(Mann-Henn-Lannes-Schwenk, Milgram, Madsen-Milgram)

Rel between $H^*(\Sigma_\infty)$ and $H^*(\Sigma_n)$ recently studied by Feshbach

$$H^*(\Sigma_4) \rightarrow H^*(\Sigma_2 \times \Sigma_2) \oplus H^*(V) \quad \text{detect.}$$

$\Sigma_4 = V \rtimes \Sigma_3$ $\overset{Sh_2(\mathbb{Z})}{\parallel}$

$$H^*(\Sigma_4) \longrightarrow H^*(\Sigma_2 \times \Sigma_2)^{2/2}$$

$$\mathbb{F}_2[\sigma_1, \sigma_2]$$

$$\hookrightarrow H^*(V)^{Sh_2(\mathbb{F}_2)} = \mathbb{F}_2[d_2, d_3] \quad \text{Dickson invariants.}$$

Using Stiefel-Whitney classes we get.

$$H^*(\Sigma_4) = \mathbb{F}_2[\sigma_1, \sigma_2, \sigma_3] / \sigma_1 \sigma_3 = 0.$$

splitting into
subgroups.

$$\sigma_1 \longmapsto (\sigma_1, 0)$$

$$\sigma_2 \longmapsto (\sigma_2, d_2)$$

$$\sigma_3 \longmapsto (0, d_3)$$

② $G = SL_3(\mathbb{F}_2)$ Use the Tits building. T .

$$T/G = \begin{array}{ccc} & D_8 & \\ \bullet & \xrightarrow{\quad} & \bullet \\ \Sigma_4 & & \Sigma_4 \end{array}$$

$$H^*(SL_3(\mathbb{F}_2) \ltimes H^*(D_8)) \cong 2 H^*(\Sigma_4)$$

$$H^*(SL_3(\mathbb{F}_2)) = \text{res}_1 H^*(\Sigma_4) \cap \text{res}_2 H^*(\Sigma_4) \leq H^*(D_8).$$

$$H^*(SL_3(\mathbb{F}_2)) = \mathbb{F}_2[u_2, v_3, w_3] / v_3 w_3 = 0$$

$$\rho = \Sigma_4 \rtimes_{D_8} \Sigma_4 \rightarrow SL_3(\mathbb{F}_2)$$

induces a mod 2 equivalence

$$B\rho \xrightarrow[\mathbb{F}_2]{\cong} BSL_3(\mathbb{F}_2).$$

$$(3) \quad G = A_6$$

$$H^*(A_6) \simeq H^*(\Sigma_6) / (\sigma_1)$$

$$\triangleq \mathbb{F}_2[\sigma_2, \sigma_3, \zeta_3] / \sigma_3 \zeta_3 = 0$$

$$\begin{array}{ccc} & B A_6 & BSL_3(\mathbb{F}) \\ \nearrow \text{mod } 2 \text{ iso} & & \nearrow \text{mod } 2 \text{ iso} \\ & B\Gamma & \end{array}$$

$$(4) \quad G = M_{11}, \quad M_{11} \subset \Sigma_{11}$$

$$|A_2(G)|/G$$

$$\begin{array}{ccc} & D_8 & \\ \bullet & \xrightarrow{\quad} & \bullet \\ \Sigma_4 & & GL_2(\mathbb{F}_3) \end{array}$$

$$H^*(M_{11}) \simeq \mathbb{F}_2[v_3, u_4, w_5] / w_5^2 + v_3^2 u_4$$

Lecture 5: Structure of group cohomology

twisted coeff R . $p \nmid |G|$.

$H^*(G)$ "compli" looks like $R = \mathbb{F}_p[\beta_1, \dots, \beta_n]$
by Noether normalization.

$U\alpha: R \rightarrow R$ has no zero divisors.

Regular sequence $\alpha_1, \dots, \alpha_r \stackrel{\text{def}}{\iff}$

$\alpha_i: R/(\alpha_1, \dots, \alpha_{i-1}) \rightarrow R/(\alpha_1, \dots, \alpha_{i+1})$ is injective $\forall i \leq r$.

$\text{depth } H^*(G) = \overset{\text{maximal}}{\text{length of regular seq in } H^*(G)}$

What can we say about depth of $H^*(G)$

$$\text{depth } H^*(G) \leq \underset{\text{Quillen}}{\text{Kulldim } H^*(G)} = r_p(G).$$

Cohen-Macaulay:
depth = dim

$H^*(G)$ is $\mathbb{F}_p$ free over poly subalg of dim $r_p(G)$

Thm (J. Carlson) If $H^*(G)$ has depth equal to r then

$H^*(G)$ is detected subgp of the form $C_G(E)$ where
 E elt ab of rank r .

Posn: $C_G(E)$ are universal detectors and we would like to understand their cohomology.

$$E \rightarrow P \quad \#kE = r \quad E \text{ central in } P$$

Thm (Duflot) The depth of $H^*(P)$ is at least r

$$\Rightarrow \text{depth } H^*(P) \geq r_P(ZP)$$

$$E \rightarrow P$$

"

$$\langle x_1, \dots, x_r \rangle$$

$$E \hookrightarrow P$$

$$\langle x_1, \dots, x_r \rangle$$

$$x_j^0 \mapsto e^{2\pi i/p}$$

$$x_1, x_2, \hat{x}_j, x_r \mapsto 1.$$

all this character X_j

$$V_j = \text{Ind}_E^P(x_j) \text{ i.e. } X \otimes_E \mathbb{C}P.$$

$$X_j = S(V_j) = \{u \in V_j \mid \|u\| = 1\}.$$

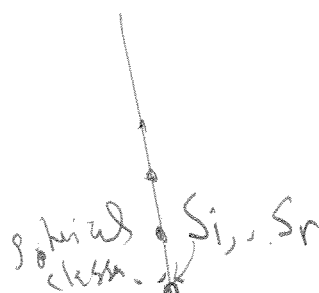
$$Z = X_1 \times \dots \times X_r$$

product of spheres with
diag action of P .

$$Z \rightarrow EP \times_P Z$$

$$\downarrow$$

BP



call $d(S_j) =: \alpha_j$
in $H^{2(p-1)}(P)$

identify with Chern
classes.

$$H^*(P)$$

Claim: $\alpha_1, \dots, \alpha_r$ form a regular sequence
in $H^*(G)$

Key fact: $V_j / \langle x_j \rangle$ then $\langle x_j \rangle$ acts without
non-zero fixedpoints.

ie $V_j / \langle x_j \rangle$ contains no copy
of the trivial character.

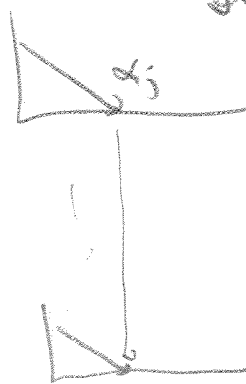
ie $S(V_j)$ has a free $\langle x_j \rangle$ -action.

Now look at comparison of S.S.

$$EP \times_P S(V_j)$$



$$E \langle x_j \rangle \times_{\langle x_j \rangle} S(V_j)$$



Behavior on
restriction deter-
mines the char-
acter

$$\text{res}_{\langle x_j \rangle}^P (\alpha_j) = \pm [P: E].$$

$$\text{where } H^*(\langle x_j \rangle) = \Lambda(\mathbb{C}) \otimes \mathbb{F}_2[t].$$

Putting this together on all the
spheres shows

$$H^*(EP \times_P Z) \cong H^*(P) / (\alpha_1, \alpha_2, \dots, \alpha_r)$$

and $\alpha_1, \dots, \alpha_r$ form regular seq

$$\langle x_j \rangle \rightarrow G \rightarrow G / \langle x_j \rangle$$

$$E_2 = H^*(G / \langle x_j \rangle) \otimes H^*(\langle x_j \rangle)$$

Special case: $rk = E = sk_p$ i.e.

even all ab subgp of P is
central (called PC gp).

In our construction E acts freely on $\mathbb{Z}$
hence P acts freely on $\mathbb{Z}$.

$$H^*(\mathbb{Z}/P) \cong H^*(P)/(x_1, \dots, x_r)$$

← Cox oriented
manifold.

i.e. it is a Poincaré
duality algebra.

One can ^{use this to} construct a ~~non~~ cohom class $\beta \in H^*(P)$
s.t. $\text{res}_K^P(\beta)$ for all proper subgps $K < P$.

Thm $H^*(G)$ is cohom-macaulay + undetectable

$\Leftrightarrow G$ is a p -gp which is PC.

Basic idea: $0 \neq \beta = \text{tr}_E^P(w)$

$$\text{res}_K^P \text{tr}_E^P w = \sum_{\substack{1 \leq i \leq [P:K] \\ \text{Eulid}}} \text{tr}_{\text{res}_{EK}^E}^{\text{res}_K^P} w$$

Eulid

$$= [P:K] \cdot \text{something} = 0.$$

KSP index 2 or rank

$$\begin{array}{ccccccc}
 H^n(P) & \xrightarrow{m} & H^n(K) & \rightarrow & H^n(P) & \xrightarrow{\quad} & H^{n+1}(P) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^n_P(Z) & \rightarrow & H^n_K(Z) & \rightarrow & H^n_P(Z) & \rightarrow & H^{n+1}_P(Z) \\
 \text{"} & & \text{"} & & \text{"} & & \text{"} \\
 H^n(Z/P) & \rightarrow & H^n(Z/K) & \rightarrow & H^n(Z/P) & \rightarrow & H^{n+1}_P(Z) \\
 \text{invol } b_1 & & \mathbb{F}_2 \rightarrow \mathbb{F}_2[P/K] & \rightarrow & \mathbb{F}_2 & &
 \end{array}$$

$$n = \dim Z/P$$

Σ

$$\begin{array}{ccc}
 H^n(E) & \xrightarrow{tr} & H^n(P) \quad \beta = \text{tr}_E^P(w) \\
 \downarrow & & \downarrow \\
 H^n(Z/E) & \xrightarrow{tr} & H^n(Z/P)
 \end{array}$$

$\mu_{Z/P} \neq 0$.
orientation
class
so $\beta \neq 0$.

Example:

① Q_8

$$H^*(Q_8)/(\text{un}) \cong H^*(SU(2)/Q_8)$$

$\text{rk } E = n$.

② Universal central ext $1 \rightarrow H_2 E \rightarrow P \rightarrow E \rightarrow 1$.

$H^*(P)$ can be computed for $p > \binom{n}{2} - 1$

$$(\mathbb{Z}_1, \dots, \mathbb{Z}_{\binom{n+1}{2}}) \rightarrow H^*(P) \rightarrow X \rightarrow 0$$

$$X = \text{Tor}_{\mathbb{F}_p[C_{ij}]}(N(e_1, \dots, e_n), \mathbb{F}_p)$$

$C_{ij} \leftrightarrow e_i e_j$
Some SIS appear at E_3 .

$$(3) \quad M_{11} \quad H^*(M_{11}) \cong \mathbb{F}_2[u_3, v_4, w_5]$$

$$/ w_5^2 \neq u_3^2 v_4$$

$$\text{as mod } \mathbb{F}_2[u_3, v_4] (1, w_5)$$

$$\text{Detected on } C_{M_{11}}(2/2 \times 2/2) = 2/2 \times 2/2.$$

$$H^*(M_{11}) \rightarrow H^*(SO_{16}) \rightarrow H^*(2/2 \times 2/2)$$

detects (should class be so by Carlson's thm)

$$(4) \quad L_p \text{ extra-special gp of order } p^3 \quad L_r = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

and exponent p .

$$H^*(L_p) = \begin{cases} \text{Cohen-Macaulay } p=3. \\ \quad + \text{ detected on proper subgroups } \\ \quad \quad \quad \cong 2/p \times 2/p \\ \\ \text{not Cohen-Macaulay} \\ \text{and has } \underline{\text{undetectable elements}} \\ \quad \quad \quad (\text{by thm since not } p < pp) \end{cases}$$