Allen

Stable Homology by the Scanning Method

Madsen-Weiss, Galatius theorems, sequels

0-dimensional case: Thm of Barratt-Priddy-Quillen

$$H_*(\Sigma_n) \sim H_*(\Omega^\infty_0 S^\infty)$$

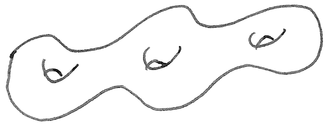
$$\begin{array}{c} \parallel \\ H_*(B\Sigma_n) \end{array}$$

2-dimensional case: Madsen-Weiss

1-Dim : Galatius : $H_*(Aut(F_n)) \sim H_*(\Omega^\infty_0 S^\infty) = H_*(\Sigma_n)$

$$\begin{array}{c} \parallel \\ H_*(BAut(\underbrace{F_n}_{\text{space of graphs}})) \end{array}$$

3-dimensional : special classes of 3-mfds



- V_g = genus g handlebody

$$H_*(BDiff(V_g)) \sim H_*(\Omega^\infty_0 S^\infty(BSO(3)_+))$$

$$\begin{array}{c} \parallel \\ H_*(\pi_0 Diff(V_g)) \end{array}$$

(same as one would get for all 3-mfds with bdry)

- $H_*(BDiff(\underbrace{\#_g S^1 \times S^2}_{\text{bdry of a 4-dim handlebody}})) \sim H_*(\Omega^\infty_0 S^\infty(BSO(4)_+))$

SCANNING : SEGAL 1979 (CONFIGURATION OF POINTS)

Σ_n = SYMM. GROUP $\quad \Sigma_n \subset \Sigma_{n+1}$, $\Sigma_\infty = \bigcup_n \Sigma_n$

$\Omega^n S^n$ = SPACE OF MAPS $(S^n, *) \to (S^n, *)$

$\Omega^n S^n \subset \Omega^{n+1} S^{n+1}$ BY SUSPENSION

$\Omega^\infty S^\infty = \bigcup_n \Omega^n S^n$

$\pi_i(\Omega^\infty S^\infty) = \pi_i^S = \pi_{i+n}(S^n)$
$\llcorner$ n LARGE

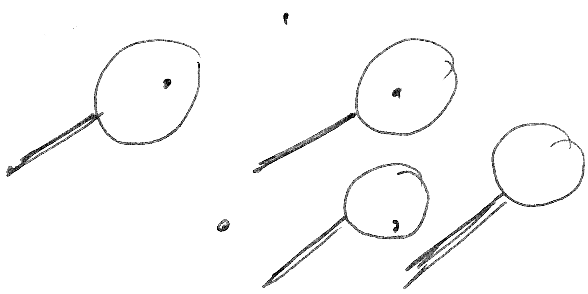
$\pi_0(\Omega^\infty S^\infty) = \pi_n S^n = \mathbb{Z}$

ALL COMPONENTS ARE HTPY EQUIV.

$H_*(\Sigma_k) = H_*(B\Sigma_k)$

MODEL FOR $B\Sigma_k$: SPACE OF UNORDERED DISTINCT k-TUPLES OF POINTS IN $\mathbb{R}^\infty = C_k(\mathbb{R}^\infty) = \bigcup_n C_k(\mathbb{R}^n)$

SCANNING : TAKE A CONFIGURATION $c \in C_k(\mathbb{R}^n)$



POWERFUL MAGNIFYING LENS WITH LIMITED VIEW.
— SEE AT MOST ONE POINT AT A TIME.

POINT IN A BALL B^n, DISAPPEARS AT ∂B^n.

$\rightsquigarrow$ GET A POINT IN S^n, FOR EACH POSITION OF THE LENS.

HENCE $\mathbb{R}^n \to S^n$, TAKING ∞ TO ∞

i.e. $(S^n, *) \to (S^n, *) \in \Omega^n S^n$.

$\rightsquigarrow C_k(\mathbb{R}^n) \to \Omega^n S^n$.

$n \to \infty$ $\quad\leadsto\quad$ $C_k(\mathbb{R}^\infty) \longrightarrow \Omega^\infty S^\infty$

$C_k(\mathbb{R}^\infty) = B\Sigma_k$ CONNECTED

LANDS IN kth ($-k$?) COMPONENT.

$k \to \infty$: $B\Sigma_\infty \longrightarrow \Omega_0^\infty S^\infty$

(ADDING PT AT ∞)

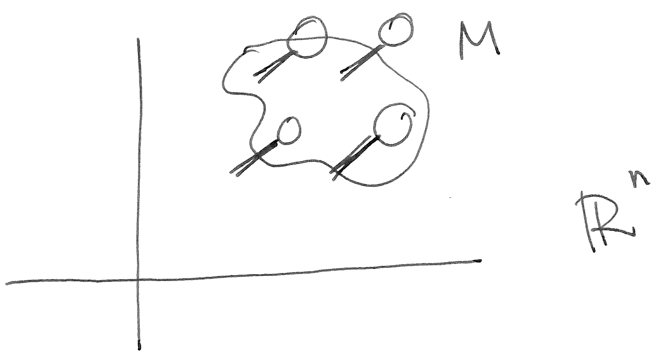
THM [BPQ] THIS INDUCES H_*-ISO

SCANNING MANIFOLDS: GIVEN A SMOOTH, CLOSED, ORIENT. MFD M OF dim m.

MODEL FOR $B\mathrm{Diff}(M)$: SPACE OF SMOOTH ORIENTED SUBMFDS OF $\mathbb{R}^\infty$ DIFFEO TO $M = C(M)$

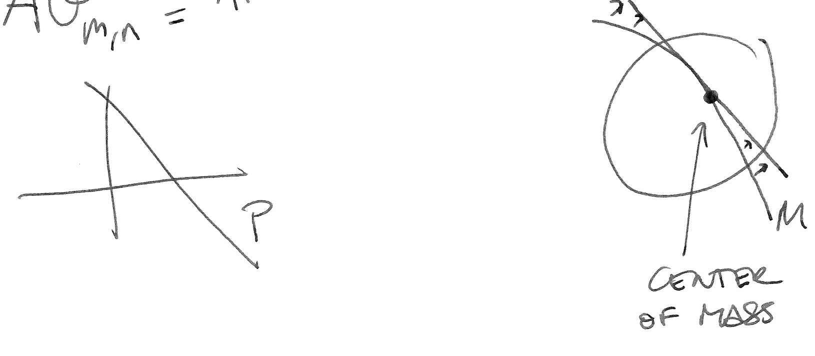
$C_n(M)$ = SUBMFDS IN $\mathbb{R}^n \subset \mathbb{R}^\infty$

SCAN A CONFIG IN $C_n(M)$:



SEE A SMALL PIECE OF M, ALMOST FLAT m-PLANE IN A BALL B^n.

$AG_{m,n}$ = AFFINE GRASSMANNIAN OF m-PLANES IN $\mathbb{R}^n$ $\simeq$ ALMOST FLAT PLANES



$C_n(M) \longrightarrow \Omega^n AG^+_{m,n}$ ← ONE-PT COMPACTIFICATION BY n-PLANE AT ∞.

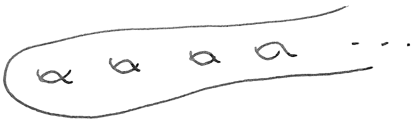
$AG^+_{m,n} \hookrightarrow \Omega AG^+_{m,n+1}$

TRANSLATE FROM $-\infty$ TO $+\infty$ IN NEW DIRECTION

LET $n \to \infty$ $\quad C(M) \longrightarrow \Omega^\infty AG^+_{m,\infty}$

$C(M) = B\mathrm{Diff}(M)$

LET M VARY.

LET $m=2$.  S_∞ INFINITE GENUS

$B\mathrm{Diff}_c(S_\infty) \longrightarrow \Omega^\infty AG^+_{2,\infty}$

COMPACTLY SUPPORTED

MW: THIS IS AN ISO ON H_*.

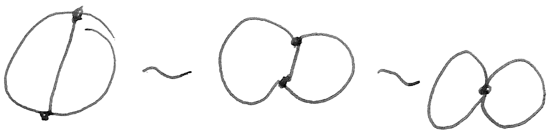
EARLE-EELLS: $\mathrm{Diff}_c(S_\infty)$ HAS CONTRACTIBLE COMPONENTS

CAN REPLACE BY $\pi_0 \mathrm{Diff}_c(S_\infty)$.

HARER STABILITY: $H_i(\pi_0 \mathrm{Diff}(S_g))$ is INDEP OF g FOR $g \gg i$.

TWO POSSIBLE DEF OF THE TOPOLOGY.

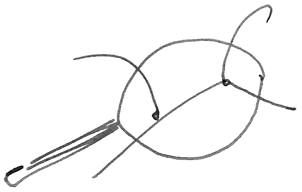
$\mathrm{Aut}(F_n)$: MODEL FOR $B\mathrm{Aut}(F_k)$: SPACE OF FINITE CONNECTED BASEPOINTED GRAPHS in $\mathbb{R}^\infty$ with $\pi_1 \cong F_k$.

ALLOW EDGES TO COLLAPSE. 

(ALLOW VALENCE 1 VERTICES)

SMOOTH EDGES

[CULLER-VOGTMANN, IGUSA]

SCAN GRAPHS IN $\mathbb{R}^n$:  SEE TREES IN A BALL B^n. (OR $\emptyset$)

SPACE OF TREES IN $B^n \simeq$ SPACE OF POINTS $= S^n$

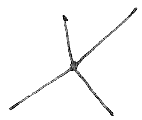
(BY SHRINKING)

$$\rightsquigarrow \quad BAut(F_g) \longrightarrow \Omega_0^\infty S^\infty$$

$$H_*(Aut\, F_\infty) \cong H_* \Omega_0^\infty S^\infty \simeq H_* \Sigma_\infty$$

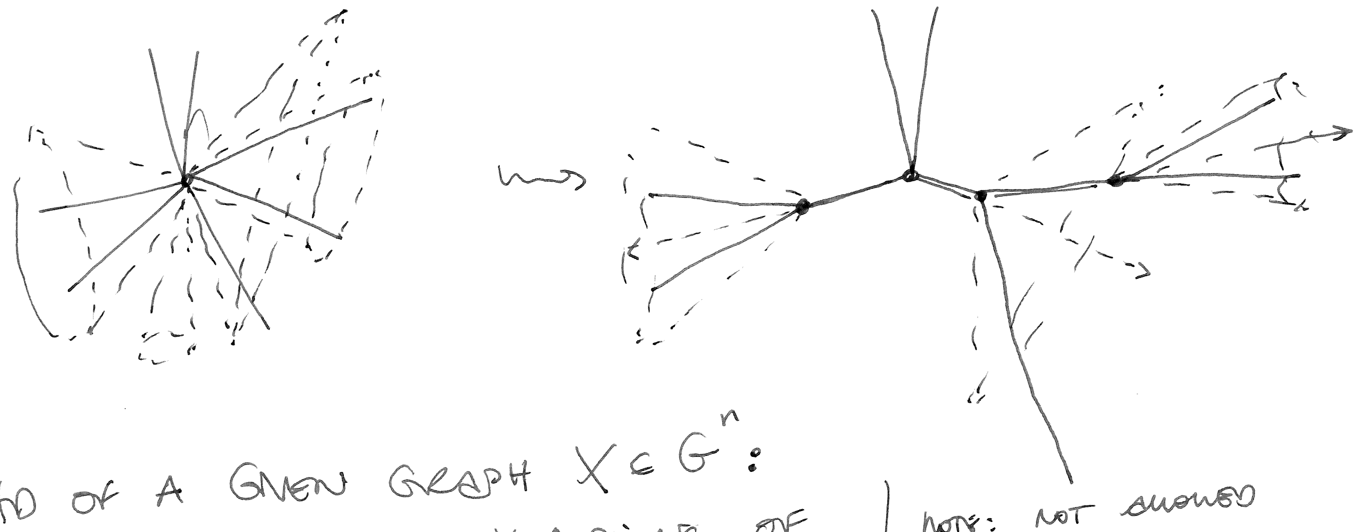
PART II : GALATIUS' THM

SPACE G^n OF GRAPHS IN $\mathbb{R}^n$: SMOOTH EDGES, LINEAR NEAR VERTICES, VALENCE 0, 1, 2 ALLOWED,



$\emptyset$ ALLOWED, NOT NECESSARILY COMPACT, CONNECTED. EDGES CAN EXTEND TO ∞.

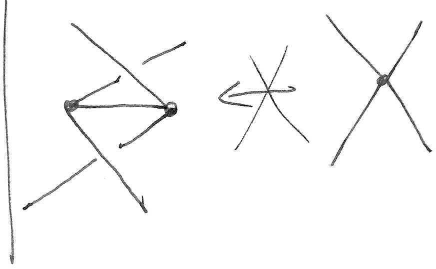
CONICAL COLLAPSING/EXPANSION OF EDGES:



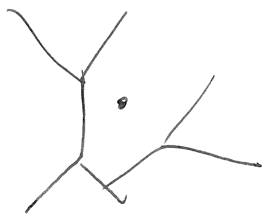
NBHD OF A GIVEN GRAPH $X \in G^n$: ALL SMALL CONICAL EXPANSIONS OF VERTICES, AND ISOTOPIES

GLOBALLY: "COMPACT-OPEN TOPOLOGY": TAKE ALL GRAPHS X' WHICH IN SOME (LARGE) BALL B^n (CENTERED AT 0) ARE CLOSE TO X. WITH $X \pitchfork \partial B^n$

NOTE: NOT ALLOWED

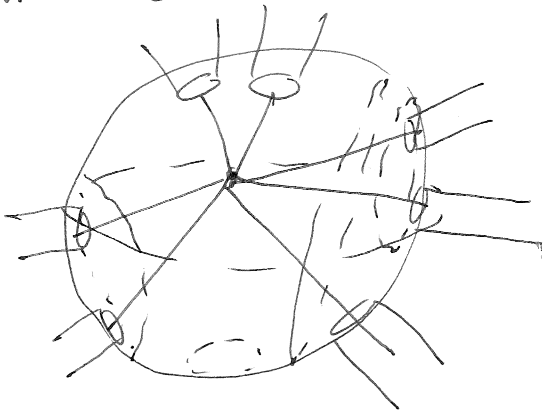


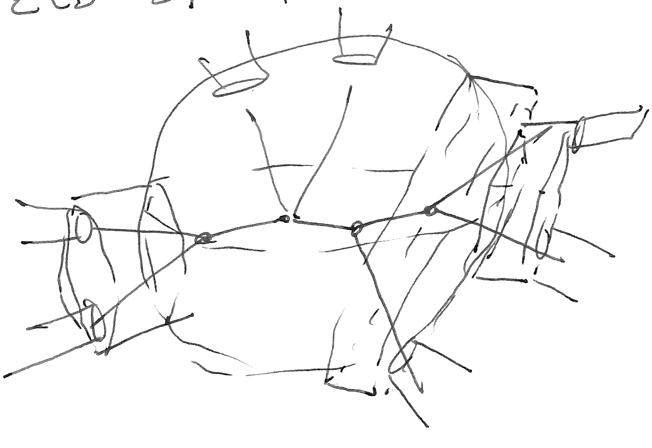
$\pi_0 G^n = 0 \quad (n \geq 1)$:



EXPAND RADIALLY FROM A POINT IN $\mathbb{R}^n - X$.

CONICAL EXPANSION IS REALIZED BY HANDLEBODIES:





0-HANDLE: BALL TRUNCATED BY DISJOINT FLAT DISCS

1-HANDLE: ROUND CYLINDER

<u>GALATIUS FILTRATION OF G^n</u>: $G^{n,0} \subset G^{n,1} \subset G^{n,2} \subset \cdots \subset G^{n,n} = G^n$

$G^{n,k}$ = GRAPHS IN $\mathbb{R}^k \times I^{n-k}$ — INFINITE IN k DIRECTIONS

<u>4 MAIN STEPS</u>

① $\pi_0 G^{n,0}$ = HOMOTOPY TYPES OF FINITE GRAPHS (POSSIBLY $\emptyset$)

LET $G^{n,0}_k$ = CONN. GRAPHS OF RANK k, CONTAINING 0 AS A VERTEX.

<u>PROP</u>: $G^{\infty,0}_k$ IS A $BAut\, F_k$

[IGUSA]: CLASSIFYING SPACE OF THE CAT OF FINITE RANK k GRAPHS, WITH MORPHISMS FORESTS COLLAPSES AND ISOMORPHISMS, IS A $BAut\, F_k$.

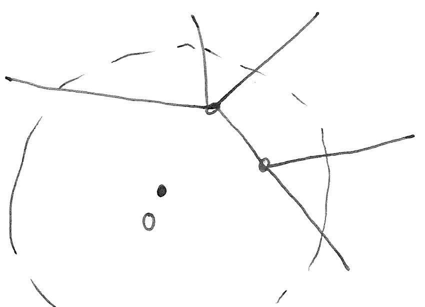
② HAVE INSIDE G^n THE GRAPHS WITH ≤ 1 POINT, FORMING AN S^n.

PROP: $S^n \hookrightarrow G^n$ IS A HOMOTOPY-EQUIV.

PROOF SKETCH: SHOW $\pi_i(G^n, S^n) = 0 \quad \forall i$

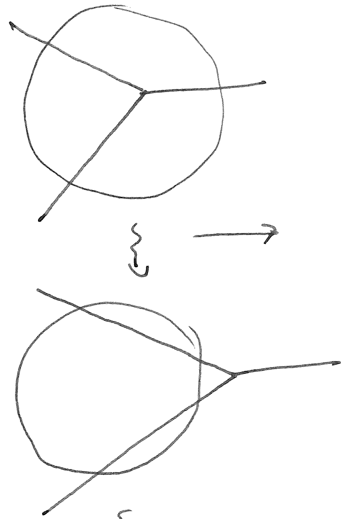
$X_t \in G^n, \quad t \in D^i, \quad X_t \in S^n, \quad t \in \partial D^i.$

WANT $B_t^n \cap X_t$ = TREE OR $\emptyset$



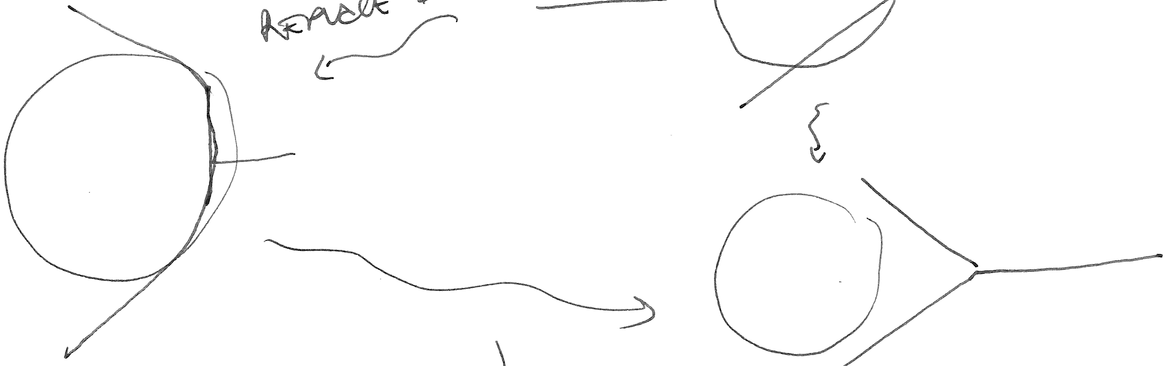
B_t^n

DIFFICULTY:



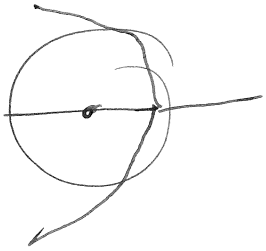
NO GOOD!

REPLACE BY



(MODIFICATION OF THE ELEMENT OF $\pi_i(G^n, S^n)$)

DON'T DO ANYTHING ON EDGES THROUGH 0?



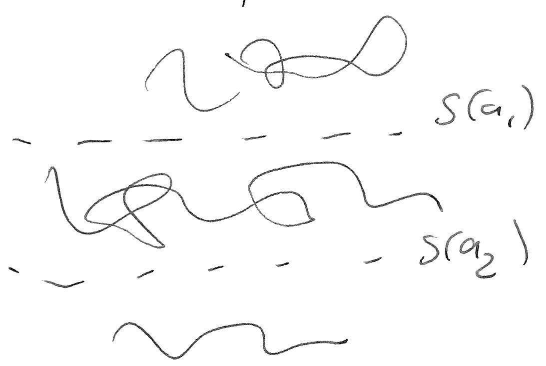
THEN EXPAND B_t^n TO $\mathbb{R}^n$ AND SHRINK EACH TREE TO ITS CENTER POINT.

(3) MAP $G^{n,\ell} \longrightarrow \Omega G^{n,\ell+1}$ BY TRANSLATING GRAPHS FROM $-\infty$ TO $+\infty$ IN $(\ell+1)$st COORDINATE.

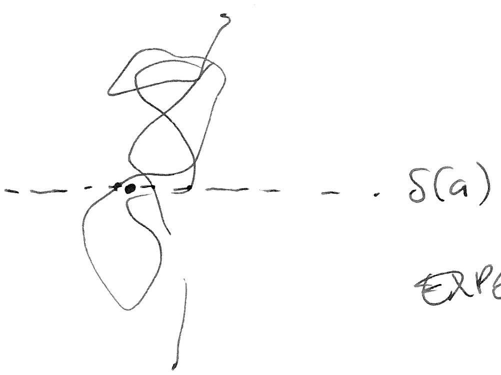
PROP: $G^{n,\ell} \longrightarrow \Omega G^{n,\ell+1}$ IS A HOMOTOPY EQUIV. $\ell \geq 1$

HENCE $G^{n,1} \simeq \Omega^{n-1} G^{n} = \Omega^{n-1} S^{n}$

IDEA: DEFINE $G_s^{n,\ell+1}$ = SPACE OF GRAPHS X IN $G^{n,\ell+1}$ WITH SLICES $\delta(a_i) = \mathbb{R}^{\ell} \times \{a_i\} \times I^{n-\ell-1}$ DISJOINT FROM X, WITH WEIGHTS $w_i \geq 0$, $\Sigma w_i = 1$.

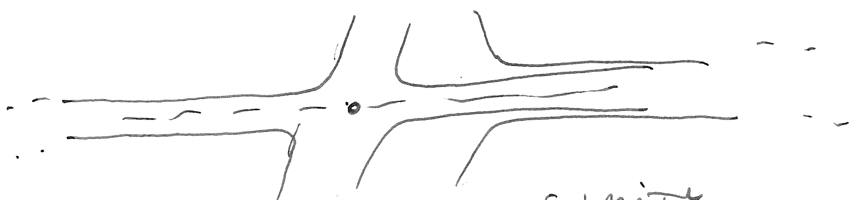


$G_s^{n,\ell+1} \longrightarrow G^{n,\ell+1}$ IS A HOMOTOPY EQUIV.



$\exists\, x \in \delta(a) \setminus G$

EXPAND RADIALLY FROM x.



FOR FAMILIES, USE A PARTITION OF UNITY — THE WEIGHTS ARE "DAMPING FACTORS" —

MONOID VERSION OF $G^{n,\ell}$: BY "STACKING AND COMPRESSION". (OR MOORE LOOPS-LIKE CONSTRUCTION)

$$G^{n,\ell} \simeq M^{n,\ell}$$

$$BM^{n,\ell} \xleftarrow{\simeq} G_s^{n,\ell+1}$$

$$\text{So } \Omega G^{n,\ell+1} \simeq \Omega BM^{n,\ell} \underset{\substack{\uparrow \\ \pi_0 M^{n,\ell} = 0}}{\simeq} M^{n,\ell} \simeq G^{n,\ell}$$

3rd Talk

G^n = SPACE OF GRAPHS IN $\mathbb{R}^n$.

$G^{n,0} \subset G^{n,1} \subset \dots \subset G^{n,n} = G^n$

$G^{n,0}_k$ CONNECTED, RANK k

① $G^{\infty,0}_\ell = BAut(F_\ell)$

② $G^n \simeq S^n$

③ $G^{n,\ell} \simeq \Omega G^{n,\ell+1}$ $\quad k \geq 1$

(FOLLOWING G-R-W) ④ $G^{n,0} \longrightarrow \Omega G^{n,1}$ NOT A HOMOTOPY EQUIV

$\pi_0 G^{n,0}$ = HOMOTOPY TYPES OF FINITE GRAPHS $(n \geq 4)$, MONOID (WITH $\amalg$

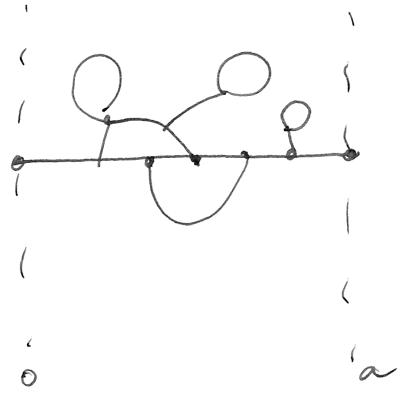
$\pi_0(\Omega G^{n,1}) = \pi_1 G^{n,1}$ GROUP

$\pi_1 G^{n,0}$ NOT ABELIAN, $\pi_1(\Omega G^{n,1}) = \pi_2 G^{n,1}$ ABELIAN.

Take $n=\infty$. Better $M^{\infty,0} = M^{\infty}$ = connected graphs in $[0,a]\times\mathbb{R}^{\infty}$ containing $[0,a]\times 0$ (baseline), meeting $0\times\mathbb{R}^{\infty}$ and $a\times\mathbb{R}^{\infty}$ in one point.

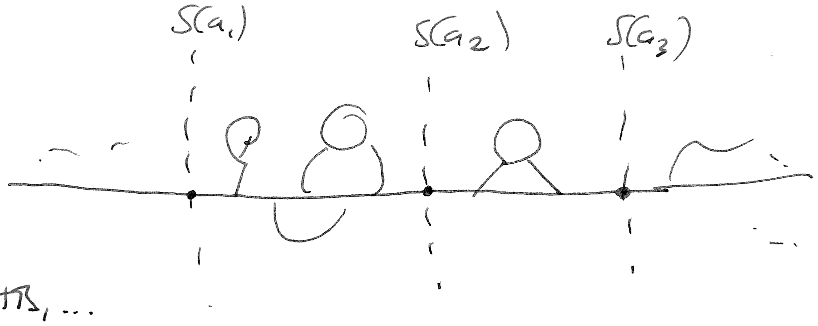
(Allow $a=0$).

$\rightsquigarrow$ Monoid.



Prop: $BM^{\infty} \simeq G^{\infty,1}$

$G^{\infty,1}_{\delta}$ = graphs containing $\mathbb{R}\times\{0\}$ with slices, weights, ...



$G^{\infty,1}_{\delta} \xrightarrow{\simeq} BM^{\infty}$

forgetting the infinite parts to left and right & slide off to ∞

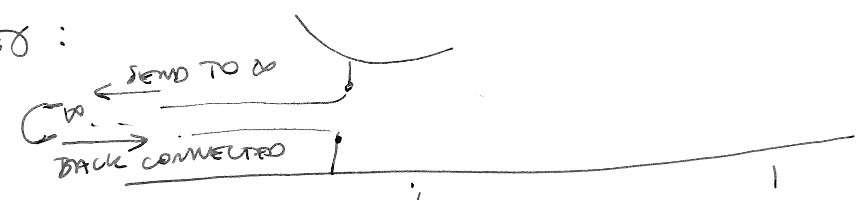
$\swarrow$

$G^{\infty,1}$

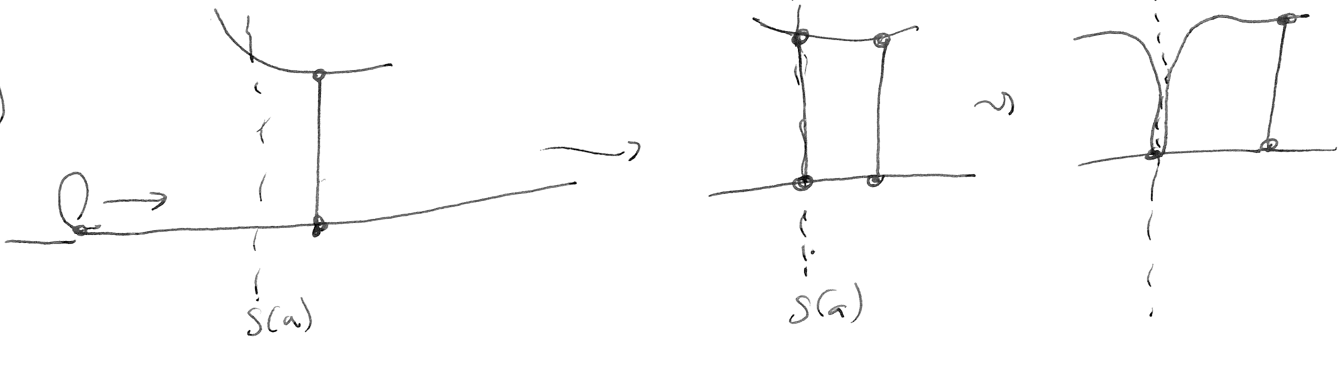
Homotopy equiv:

(a) Introduce baseline

(b) Connecting ties to baseline



(c)



ALL TOGETHER:

COMP OF M^∞ CONSISTING OF RANK ℓ GRAPHS

$$H_*(\lim_\ell BAutF_\ell) \simeq H_*(\lim_\ell M^\infty_\ell)$$ BY A VARIANT OF 1.

$$\simeq H_*(\Omega_0 BM^\infty)$$ GROUP COMPL. THM (M^∞ IS HOMOTOPY COMM.)

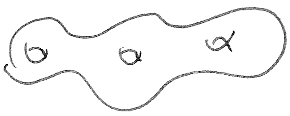
$$\simeq H_*(\Omega_0 G^{\infty,1})$$ BY 4

$$\simeq H_*(\lim_n (\Omega_0 G^{n,1}))$$

$$\simeq H_*(\lim_n \Omega_0^n G^n)$$ BY 3

$$\simeq H_*(\lim_n (\Omega_0^n S^n))$$ BY 2

$$\simeq H_*(\Omega_0^\infty S^\infty).$$

- VERSION FOR 0-DIM MFDS ⇝ $H^*(\Sigma_\infty) \simeq H^*(\Omega_0^\infty S^\infty)$
- VARIANT: PARTICLE WITH $\mathbb{Z}$-LABELS; COLLIDING ⇝ ADDITION
  ⇝ DOLD-THM: FREE AB GP GEN. BY $S^n = \mathbb{R}^n \cup \infty$ (BASEPOINT = IDENTITY)
  $\simeq K(\mathbb{Z}, n)$.
- HANDLEBODY V_g 

$$\pi_0 \mathrm{Diff}(V_g) \hookrightarrow \pi_0 \mathrm{Diff}(\partial V_g)$$ INFINITE INDEX

$\downarrow$ ONTO (HUGE KERNEL)

$Out(F_g)$

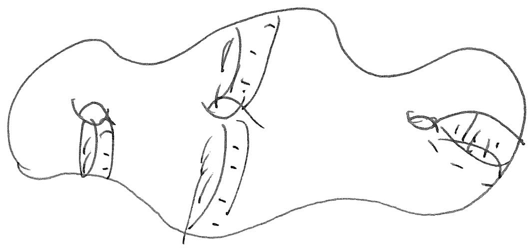
FACT: COMPONENTS OF $\mathrm{Diff}\, V_g$ ARE CONTRACTIBLE ($g \geq 2$).

$H_*(B\mathrm{Diff}(V_g))$ STABILIZER

VERY SIMILAR PROGRESS TO SHOW $H_*(\lim \pi_0 \mathrm{Diff}(V_g, D))$
$\cong H_*(\Omega_0^\infty S^\infty(BSO(3)_+))$

MAIN DIFFERENCE IN ① AND ②

① $B\mathrm{Diff}(V_g)$ = SPACE OF $V_g \subset \mathbb{R}^\infty$
WITH HANDLE DECOMPOSITION



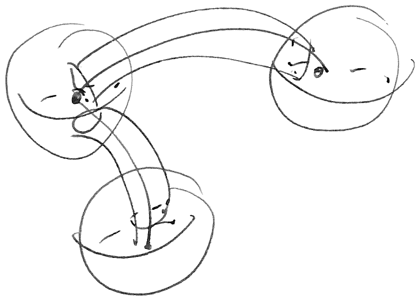
COLLECTION OF DISJOINT DISCS CUTTING THE HANDLEBODY INTO DISJOINT BALLS.

(THE DISCS HAVE WEIGHTS AND MOVE AROUND ...)

→ THICKEN TO SMALL HANDLES.

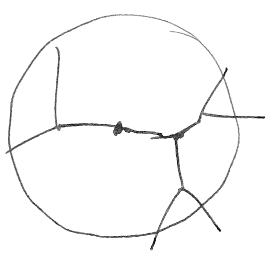
MAKE THE 2-DISCS (1-HANDLES) FLAT AND ROUND (USING $\mathrm{Diff}(D^2) \simeq O(3)$)
3-DISCS (0-HANDLES) ROUND (___ $- D^3 - O(3)$)

$\rightsquigarrow$

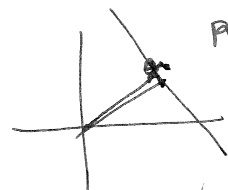


3-PLANE GRAPH
$X \subset \mathbb{R}^\infty$
FIELD OF 3-PLANES, INCLUDING TANGENT LINES TO EDGES.

②



SHRINK TO CENTER POINT $\rightsquigarrow$



POINTED 3-PLANE

(THE PLANE DISAPPEAR WHEN THE POINT → ∞)

→ THOM SPACE ($T+N \to G_{3,\infty}$)
$\simeq \Sigma^n(BSO(3)_+)$

(GET HALF THE MMM-CLASSES)

FOR $\#_g S^1 \times S^2$ SAME WITH THE COMPLEX OF SPHERES INSTEAD OF DISCS ...

(STABILITY NOT KNOWN.)