PROBLEM SESSION ON P-LOCAL FINITE GROUPS – COPENHAGEN 24/7-2007

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1. Classification of fusion systems

Background: At this time there is only one known exotic family of fusion systems at $p = 2$, the Solomon fusion system $Sol(q)$ considered by Solomon, Benson, Broto-Oliver, and Aschbacher-Chermak. At odd primes exotic examples seem to be more plentiful (Broto-Levi-Oliver, Ruiz-Viruel, Ruiz,...).

- Can fusion theorems (at $p = 2$ or all primes) be classified? Is $Sol(q)$ the only exotic family at $p = 2$?
- Use structure of proof of classification to make intelligent search (i.e. use classification experience) for exotic examples. Many exotic examples for p odd satisfies $O_p(\mathcal{F}_0) \neq 1$, where $\mathcal{F}_0 = \langle N_{\mathcal{F}}(Q) \mid Q \text{ essential}, Q \trianglelefteq S\rangle$, although this is not always true, eg in the Ruiz-Viruel examples. Are exotic fusion systems with such $\mathcal{F}_0$ somehow the most generic?
- Is there a dichotomy between small and large rank (where rank eg means the number of essential subgroups of $\mathcal{F}$). Is large *rank* better behaved? (eg is it possible to rule out exotic fusion systems of large rank at $p = 2$?) Are large *primes* better behaved in some ways (cf Ruiz-Viruel)?
- For every finite group G, is there another finite group H such that $\mathcal{L}_p(G) \cong \mathcal{L}_p(H)$ and $\mathrm{Out}(H) \to \mathrm{Out}(\mathcal{L}_p(H))$ is split surjective?

2. Classification of p-local finite groups

Background: There is an extensive literature on the higher limits obstruction theory of passing from a fusion system to a p-local finite group involving higher derived functors of the inverse limit functor (cf. Jackowski-McClure-Oliver, Oliver, Grodal,...). Using the classification of finite simple groups, Oliver proved that every fusion system coming from a finite group gives rise to a *unique* p-local finite group. This is also easy to verify by concrete calculations for all *known* exotic fusion systems.

- Prove the "generalized Martino-Priddy conjecture" that there is a unique p-local finite group associated to every fusion system (preferably a proof which does not rely on classification results).

3. Classification of finite simple groups using a classification of fusion systems

Background: The classification of finite simple group often proceeds by classifying local structures, and then showing that there exists a finite group with that structure and it is unique. The hope is that some of these arguments can be simplified using p-local finite groups.

- For G a finite simple group, when is the fusion system $\mathcal{F}(G)$ simple? Is it possible to list the G where this is not the case? (startup problem)
- Does the classification naturally divide up in the characteristic p case and the non-characteristic p case? (characteristic p means that for all non-trivial subgroups Q of $\mathcal{F}$, the fusion system $N_{\mathcal{F}}(Q)$ is constrained). The signalizer functor problems seem to be restricted to the non-characteristic p case?
- In characteristic p, understand the Meierfrankenfeld program in terms of p-local finite groups (see Meierfrankenfeld's web page http://www.mth.msu.edu/∼meier/ for more information).
- If $\mathcal{F}$ is a simple characteristic p fusion system of rank say ≥ 3, is $\pi_1(\mathcal{L})$ a finite group? Is $\mathcal{F}_p(\pi_1(\mathcal{L})) = \mathcal{F}$? Grodal-Oliver have proved that $\pi_1(\mathcal{L}_p(G)) = G$ in many cases, and so far, the examples where one doesn't get G back violate one of the above assumptions.
- How can one deal with the signalizer functor problem in the non-characteristic p-case? (the next section also addresses this question).
- Aschbacher has a program for dealing with the non-characteristic p case. See his slides from the Copenhagen 2007 meeting for more information.

4. Infinite groups realizing fusion systems

Background: Robinson and Leary-Stancu have showed how to realize any fusion system inside a (usually infinite) group. Robinson's construction proceeds by using results of Broto-Castellana-Grodal-Levi-Oliver to construct all the local subgroups $N_{\mathcal{L}}(Q)$ for Q $\mathcal{F}$-centric from $\mathcal{F}$, and then, instead of trying to piece these together to form $\mathcal{L}$, forms an iterated amalgam over their Sylow intersections.

- Can we also always realize linking systems inside a possibly infinite group? The Aschbacher-Chermak paper can be seen as doing this for $Sol(q)$.
- Furthermore, can the normalizers of centric radical subgroups be chosen finite? It is not finite for the Aschbacher-Chermak amalgam and it may be that the methods of the Chermak-Oliver-Shpektorov paper show that this is indeed impossible for $Sol(q)$? If this is not always possible for the exotic fusion systems, what is the precise obstruction to this being possible?
- What about transporter systems? Is there always some sort of large "universal" transporter system $\mathcal{T}$ associated to $\mathcal{L}$?
- Examine the obstruction theory for passing between a transporter system $\mathcal{T}$ and a linking system $\mathcal{L}$. Much less is known than in the case of passing between $\mathcal{F}$ and $\mathcal{L}$.
- Given a p-local finite group, what is the largest collection of p-subgroups which support a transporter system? If Q is an object in a transporter system, then $C_{\mathcal{L}}(Q)$ has to come from a finite group. Is the converse true?
- Is there an obstruction theory for adding objects to a transporter system?
- Assume $\mathcal{F}$ is a fusion system of rank at least 3. In the cases where $\pi_1(\mathcal{L})$ does not give you a suitable group back (e.g. Ly or $Sol(q)$), can this be remedied by passing to a transporter system $\mathcal{T}$? What is the "signalizer functor obstruction" for doing this?

5. Relationship to conjectures of Alperin/Broue/...

(Robinson) Let B be a block with defect group P and $\mathcal{F}$ the fusion system coming from B-subpairs. Let G be the iterated amalgam constructed by G. Robinson using all $\mathcal{F}$-centric radical subgroups. Does one of the following two cases always occur:

(1) $O_p(\mathcal{F}) \neq 1$, or
(2) There exists a free subgroup $N \trianglelefteq G$ of finite index such that $H = G/N$ has a p'-central extension $\widetilde{H}$ having a block $\widetilde{B}$ with defect group $\widetilde{P} \cong P$ and such that there is a defect preserving bijection between the complex irreducible characters of B and the complex irreducible characters of $\widetilde{B}$ (in general such that they have the same block theoretic invariants)?

The point of this would be that in case (1) Clifford theory can be used to reduce to a "lower rank" situation. In case (2) the representation theory of $\widetilde{H}$ is built from p-local information, hence giving a reason why the standard block theoretic invariants should be p-locally determined.

6. Relationship to loop spaces

(Benson) For a finite group G, D. Benson proves that for $A = ekGe$, $H_n(\Omega(BG\hat{_p}); \mathbb{F}_p) \cong \mathrm{Tor}^A_{n-1}(kGe, ekG)$, $n \geq 2$, where $e = 1 - f$ for f an idempotent corresponding to the trivial module.

- Can one do something similar for p-local finite groups? Maybe one can take A to be an algebra constructed from the twisted category algebra coming up in Linckelmanns work?
- Find conditions expressed in terms of just the p-local finite group for polynomial growth of the loop space on BG p-completed.